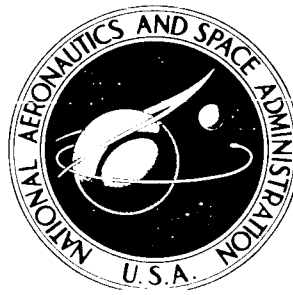**NASA CONTRACTOR REPORT**

NASA CR-55

N64-22041

SYNTHESIS OF A SIMPLE SHOCK ISOLATOR

by Lucien A. Schmit, Jr., and Richard L. Fox

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • JUNE 1964

SYNTHESIS OF A SIMPLE SHOCK ISOLATOR

By Lucien A. Schmit, Jr., and Richard L. Fox

NATIONAL AERONAUTICS AND SPACE ADMINISTRATION

SYNTHESIS OF A SIMPLE SHOCK ISOLATOR

By Lucien A. Schmit, Jr. and Richard L. Fox

Case Institute of Technology
Cleveland, Ohio

SUMMARY

2 2 0 4 1

The simple spring-mass-damper system shown in Fig. 1 is considered as a shock isolating system. The problem is to determine the spring stiffness, k, and the damping coefficient, c, for optimum performance. Specifically this note reports the successful development of a capability of solving both of the following two problems:

1) The "rattle space" is limited by external considerations and the problem is to choose the spring-damper combination such that the maximum acceleration is a minimum and such that the maximum relative displacement is consistent with the "rattle space".

2) The "fragility level" of the mounted unit is known and the problem is to choose the spring-damper combination such that the maximum relative displacement is a minimum while the maximum acceleration is less than the fragility level of the unit.

The capability reported here solves these two problems when the shock environment consists of a single shock pulse or several shock pulses.

AUTHOR

INTRODUCTION

There are many design problems for which there are numerous solutions in the sense of fulfilling a given set of requirements and specifications. In these problems there is often a criterion such as weight, cost, serviceability and appearance by which one acceptable design may be judged better than another. If this merit criterion can be expressed as a function of the design parameters the selection of the best possible design becomes conceivable. In many problems this optimization may be attacked in an analytical fashion using max-min techniques. However, in a large class of problems, due to the nature

of the merit function and/or the constraints placed on the system by the design requirements, it is not feasible or not possible to use closed form analytical techniques. This may result for example from the fact that the merit "function" is a function only in the sense that it is a "rule" for determining the merit associated with a given design. In this situation the function may be thought of as a "black box" into which are put the values of the parameters representing a given design and out of which comes the value of the merit for that design. The box may contain such things as differential equations, a modal superposition analysis, an analogue computer and so on. Even if the merit function is simple, the constraining relations of the design requirements may be of a nature that precludes the use of an analytical approach.

Several problems of this type have been solved. These have been problems for which the technology has been the mechanics of deformable structures under static load[1,2,3]. This note presents the solution of an elementary problem involving a dynamics technology.

Consider a simple spring-mass-damper system with one degree of freedom as shown in Fig. 1. The base is subjected to an acceleration, $\ddot{y}(t)$ which is of finite duration and which will be referred to as a shock pulse. Upon the application of the pulse the mass will experience an acceleration $\ddot{x}(t)$ and will undergo relative displacement with respect to the base, $z(t) = y(t) - x(t)$.

The absolute maxima of these quantities $(\ddot{x}_m, z_m)$ may be considered as representative of the response, and the spring-damper combination thought of as a shock isolator. The mass may represent a unit to be protected. Two problems may now be stated.

(1) If the "rattle" space (z_m) is limited by external considerations, the problem is to choose the spring-damper combination which provides the least $\ddot{x}_m$ while resulting in a z_m consistent with the available "rattle" space.

(2) The "fragility level", or maximum endurable acceleration of the unit, is known and the problem is to choose the spring-damper combination providing the least possible value of z_m, thus, making it possible to mount the system in the smallest possible space.

It is realized that $\ddot{x}_m$ does not totally characterize the damaging capability of the shock felt by the unit since its time history is sometimes quite significant in this respect. However, in a large number of cases the sensitive elements of the unit are sufficiently rigid so that $\ddot{x}_m$ indeed tells how well it is protected.

In many cases the environment from which the package is to be protected contains several different shocks $\ddot{y}_1(t)$, $\ddot{y}_2(t)$, ..., $\ddot{y}_n(t)$, all of which may be assumed to be applied with the system at rest. For each of these there will be a z_{m_i} and a x_{m_i} from which we can define the maxima:

$$\ddot{X}_m = \max\left[\ddot{x}_{m_1}, \ddot{x}_{m_2}, ..., \ddot{x}_{m_n}\right] \qquad (1)$$

$$Z_m = \max\left[z_{m_1}, z_{m_2}, ..., z_{m_n}\right] \qquad (2)$$

With these definitions the two problems stated above remain essentially the same only now there is a multiplicity of load conditions and z_m and x_m are replaced by Z_m and X_m respectively. It should be noted that $\ddot{X}_m$ and Z_m are functions of the design parameters k and c, i.e., $\ddot{X}_m = \ddot{X}_m(k,c)$ and $Z_m = Z_m(k,c)$ (assuming the mass specified and the pulse set assigned).

In this note the examples are primarily of Type 1 in which the acceleration is to be minimized. This choice is purely arbitrary since arguments for the existence of both problems can be advanced. The choice, however, has been fortunate because one of the primary aims of the work was to discover what kinds of poorly behaved merit functions existed and to develop techniques, if possible to handle them. As will be seen later the $\ddot{X}_m(k,c)$ as a merit function is much more "pathological" than is $Z_m(k,c)$.

Much work in engineering synthesis has dealt with weight, a merit function that is independent of the system loads. The present problem has the property that $\ddot{X}_m(k,c)$ is, in general, the result of one condition (pulse) for a given k and c, but for a different k and c it may be the result of a different pulse. It was expected that this would give rise to a behavior of $\ddot{X}_m(k,c)$ that might be difficult to handle by existing techniques.

Another reason for choosing this problem was to demonstrate the feasibility of using a dynamics technology in a synthesis problem and to discover what difficulties might be inherent in such an application. Usually a dynamic analysis is considered solved if the time response of the system is obtained. However, in most cases only certain aspects of the response history are significant to the design problem. For example, as discussed above certain maxima may be of interest. In other situations, the important factor may be the number of times a given quantity exceeds some number (as in fatigue), or the length of time required for damping to a given level, and so on. In the conventional design process the response is obtained and more or less quantitative judgements are made regarding its acceptability. In a systematic synthesis these judgements must be formalized into explicit decisions.

SYMBOLS

A	a constant of integration
B_i	the magnitude of the ith square pulse
C_1, C_2	constants of integration
D	a constant of integration
G	a constant
$H(t)$	the Heaviside step function
K_1, K_2	constants of integration
N	an integer
Q	a constant
R	a constant of integration
$\bar{R}$	a vector from $\bar{V}_i$ to $\bar{V}_{i+2}$
S	a constant of integration
T_i	the duration of the ith square pulse
V	a constant
$\bar{V}_i$	a design vector; $i = 1$ represents the current design
W	a constant
$\ddot{X}_m$	the greatest of $\ddot{x}_{m_i}$
Z_m	the greatest of z_{m_i}
c	the damping coefficient
k	the spring stiffness
m	the mass
$\bar{m}_i$	a move vector
n	the damping ratio, $c/2m$

t_o the time at which a maximum occurs

x the absolute displacement of the mass

$\ddot{x}_{m_i}$ the maximum of $\ddot{x}(t)$ due to the ith pulse

y the absolute displacement of the base

$\ddot{y}_i$ the ith shock pulse

z the relative displacement, $y-x$

z_{m_i} the maximum of $z(t)$ due to the ith pulse

$z(T), \dot{z}(T)$ the values of $z(t), \dot{z}(t)$ for $t = T$, the end of the pulse

$\alpha, \beta, \gamma, \delta$ constants of integration

ω_o undamped circular frequency $\sqrt{\dfrac{k}{m}}$

ω_A damped circular frequency $\sqrt{\omega_o^2 - n^2}$

ω_{od} overdamped pseudo frequency $\sqrt{n^2 - \omega_o^2}$

$\nabla\varphi$ grad φ

$\Delta \varphi$ a finite change in φ

$\doteq$ approximately equals

SYNTHESIS FORMULATION

The Design Parameter Space

In synthesis problems it is often quite useful to think of a cartesian space, the coordinates axes of which are the design parameters (see Fig. 2). Thus, each point in the space represents a distinct design. In a given design problem there will be points which represent acceptable or feasible designs and points which are unacceptable. Some designs are ruled out because of natural or imposed limits on the design parameters themselves; for example, negative spring constants are ruled out or there may be a requirement that damping be sub-critical in a

dynamics problem, or dimensional limitations in a structural problem. These constraints, which do not involve the response of the system to the applied loads, will be called side constraints.

Some designs will be acceptable in terms of the side constraints but unacceptable in terms of performance, for example, a structural design that displaces too much or simply collapses, or a dynamics design where some response characteristic is unacceptable. These constraints will be called behavior constraints.

The design space can now be thought of as divided into two sub-sets[*]: the acceptable designs and the unacceptable designs. The surface[+] dividing them is called the composite constraint surface.

Associated with each acceptable point is a value of the merit function. For a single value of the merit function the points form a surface. The solution to the optimization problem can then be thought of as that acceptable design (or those designs) lying on the merit surface having the best merit value. In most problems solved so far, this design has been on the composite constraint surface, at a point which, in a rough way, can be thought of as a tangency point between the best merit surface and the composite constraint surface. This, however, is not necessarily the case; the best design may be at an interior point of the acceptable region in some problem.

About the Analysis

The analysis of the spring-mass-damper system is quite simple and uncomplicated for ordinary use but when it is to be used in a synthesis, some effort must be expended to get it into a usable form.[++]

---

[*] The acceptable set may be composed of several disjoint sets. It might in some instances contain one point in which case the problem is to find that point and optimization is no longer a question. The acceptable set may be a null set which, of course, means that the problem has no solution.

[+] In the case of N design parameters, the surface is the totality of points satisfying some $F(x_1, x_2, x_3 ..., x_n) = 0$ which is a sub-space C_{n-1} and divides the design space into two sub-sets: those points for which $F < 0$ and those points for which $F > 0$. In this sense it is a surface; it is not a two dimensional object but an $n - 1$ dimensional object, which may or may not be disjoint.

[++] See Appendix A for details.

The differential equation of the system is:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0$$

taking $z = (y - x)$:

$$\ddot{y} - \ddot{z} - \frac{c}{m}\dot{z} - \frac{k}{m}z = 0$$

or

$$\ddot{z} + \frac{c}{m}\dot{z} + \frac{k}{m}z = \ddot{y}(t)$$

with initial conditions:

$$z(0) = 0, \qquad \dot{z}(0) = 0$$

Corresponding to each pulse, $\ddot{y}_i(t)$ there will be a $z_i(t)$ from which $\ddot{x}_i(t)$ can be obtained.

For the actual problem only square pulses with different durations and magnitudes were used (Fig. 3). The reason for doing this instead of having several different shaped pulses was mainly the exploratory nature of the investigation. The use of the square pulses adequately illustrates the salient features of the problem.

Using the Heaviside step function, $H(t)$, the equation can be written:

$$\ddot{z}_i + \frac{c}{m}\dot{z}_i + \frac{k}{m}z_i = B_i(H(t) - H(t-T_i)),$$

$$i = 1, 2, \ldots, n$$

where B_i and T_i are the magnitude and duration of the ith pulse.

The relative displacement maxima occurs where $\dot{z}_i(t) = 0$. This is true whether the maximum occurs during the pulse, just at the end of the pulse, or after it has ceased. So finding z_{m_i} is a matter of taking the derivative of the solution for the response during and after the pulse, solving for the t_o's for which $\dot{z}_i(t_o) = 0$, substituting the t_o's into the response equations. and comparing these maxima for the maximax or greatest maximum for the ith pulse. Then in order to find Z_m the z_{m_i} must be compared (see equation (2) page 3).

The absolute acceleration, $\ddot{x}_{m_i}$ is a somewhat different matter since $\dddot{x}_i$ (t) is discontinuous at the end of the square pulse which means that the maximum acceleration may occur at the end of the pulse without having $\dddot{x}_i(T_i) = 0$. In finding the **maximum** of the acceleration, $\ddot{x}_{m_i}$, the acceleration peaks during the pulse and after the pulse must also be compared with $\ddot{x}_i(T_i)$. These also must be compared to find $\ddot{X}_m$ (see equation (1) page 3).

Synthesis

The basic method of synthesis chosen for this problem was the gradient steep descent---alternate step method. This technique is shown diagramatically in Fig. 4 and by a basic flow chart in Fig. 5.

It consists of moving from an initial acceptable point in the direction* of the gradient to a better design some finite distance away. This process is repeated until a constraint is encountered which prevents further moves in the gradient direction. Then an alternate step is taken which is a move more or less along the constant merit curve (or surface). After the alternate step a free (unconstrained) point should have been obtained from which a steep descent can be made. The process is continued until no move can be made by either mode at which time an optimum is said to be achieved+. The reasoning behind this technique is that since the gradient points in the direction of greatest change it is the best direction to move to improve the design. If a move cannot be made in the best direction then a move is made which at least does not decrease the merit of the design.

In principle this method is quite straightforward, however, it has a number of difficult points. One of the first of these to be encountered is the tacit assumption that one can move with ease along a constant merit curve. This difficulty is characterized by the fact that the merit is an involved rule for determining the merit associated with a given design. This means that moves along the merit curves require, in general, a difficult iteration process.

Such a drawback is even more severe in an alternative method known as the constrained gradient technique. Since a move in the gradient direction cannot generally be made from a bound point this method seeks to move in the next best direction, the projection of the gradient on the constraint. This method is illustrated in Fig. 6. The reason it is more

---

* In this problem in the <u>direction</u> of the gradient but in the negative <u>sense</u> since the function is to be minimized.

+ For the question of relative minima, see the example shown in Fig. 23 .

severe for this method is that the projecting must be done by iteration and the projection must be quite exact for the method to work. Also, the iteration must be carried to completion before it can be determined if the move is of a usable length or if it must be shortened (Fig. 7). A lesser degree of accuracy is required for the alternate step method as will be seen later. These difficulties, coupled with the fact that in many problems the constraint surface is more irregular than the merit surfaces, made the alternate step system seem the more promising.*

Another difficulty of the gradient-alternate step method is that with the "black box" type of function the gradient cannot be obtained in a closed analytical form. This is surmounted simply by using a finite difference method of numerically computing the gradient.

From the definition of the gradient of a function of two variables:

$$\nabla \varphi(x,y) = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \frac{\varphi(x + \Delta x, y) - \varphi(x,y)}{\Delta x} \hat{i} + \frac{\varphi(x, y + \Delta y) - \varphi(x,y)}{\Delta y} \hat{j}$$

The partial derivatives can be approximated by computing:

$$\frac{\varphi(x + \Delta x, y) - \varphi(x,y)}{\Delta x}$$

and

$$\frac{\varphi(x, y + \Delta y) - \varphi(x,y)}{\Delta y}$$

for smaller and smaller Δx and Δy until their change from the previous calculation is less than some desired amount.

---

* There is another, perhaps more serious, theoretical deficiency of the constrained gradient method as it is often stated. This is that once a section of the composite constraint surface is encountered, it may not be left. However, in general there is no assurance that the final true optimum design will lie on this constraint or any of the others as they are encountered in order. This point is often overlooked (for example see ref. 4). If the composite constraint surface can be treated as one function then the method is theoretically correct, but this is usually quite difficult, if not impossible.

There is no rigorous assurance that this process will produce the true components of the gradient but it usually does. When it does not, other steps must be taken which will be discussed subsequently.

After the components of the gradient are determined, there is still the question of how far to move in the gradient (actually the normal) direction. This is done by using the fact that:

$$\nabla\varphi \cdot \bar{dr} = d\varphi$$

where for two variables:

$$\nabla\varphi \cdot dr = \frac{\partial\varphi}{\partial x} dx + \frac{\partial\varphi}{\partial y} dy$$

Since the move is to be made in the direction of greatest decrease of φ, $\bar{dr}$ should be along the normal. This requires that

$$\frac{dy}{dx} = \frac{\frac{\partial\varphi}{\partial y}}{\frac{\partial\varphi}{\partial x}}$$

therefore, for small moves $\Delta\bar{r}$ along the normal:

$$\Delta x \doteq -\Delta\varphi \frac{\frac{\partial\varphi}{\partial x}}{(\frac{\partial\varphi}{\partial x})^2 + (\frac{\partial\varphi}{\partial y})^2}$$

$$\Delta y \doteq -\Delta\varphi \frac{\frac{\partial\varphi}{\partial y}}{(\frac{\partial\varphi}{\partial x})^2 + (\frac{\partial\varphi}{\partial y})^2}$$

where $\Delta\varphi$ is the desired decrease in φ for the move $\Delta\bar{r}$.

Since moves are made in finite steps along the gradient direction some difficulties are encountered in a gradient field that changes radically from point to point. If the move in the gradient direction yields a design point of worse merit than the point being moved from, the move can be shortened until it yields a point of better merit. This situation is illustrated in Fig. 8a. For shortening a halving process has been found to work satisfactorily.

In the "trough" situation (which actually occurs in the present problem) shown in Fig. 8a , a zigzagging of the moves may occur as in Fig. 8b . This zigzag is really not a failure of the method since it continues to trend in the correct direction, but it is extremely inefficient. It is characterized by a small angle between two successive gradient move vectors and can thus be detected by computing that angle or its cosine.

This computation can be performed quite simply by "remembering" the last three designs $\bar{V}_1$, $\bar{V}_2$, $\bar{V}_3$. The move vectors are then:

$$\bar{m}_1 = \bar{V}_2 - \bar{V}_3$$

$$\bar{m}_2 = \bar{V}_1 - \bar{V}_2$$

where $\bar{V}_1$ is the most recent design (see Fig. 9).

It was decided beforehand that an angle of less than 90° was an undesirable amount of zigzag (other angles may, of course, be selected). This criterion requires:

$$\frac{\bar{m}_1 \cdot \bar{m}_2}{|\bar{m}_1| \cdot |\bar{m}_2|} \leq 0$$

which is a simple calculation.

It can be noted in Fig. 9 that a vector from the oldest design of the three to the newest (or $\bar{R} = \bar{V}_1 - \bar{V}_3$) is roughly parallel to the gradient that would be obtained directly in the middle of the "trough". This then may be an efficient direction to move.

These ideas were incorporated into the synthesis program by first providing a test for zigzag by "remembering" designs and then testing the dot products of the move vectors. If zigzagging occurs the next move tried is $\bar{V}_1 + \bar{R}$; if this is successful (that is, if the merit

improves) then a move $\bar{V}_1 + \overline{2R}$ is tried then $\bar{V}_1 + \overline{LR}$ and so on until either the direction fails to improve the design or a constraint is encountered. Whenever the direction fails to improve the design, including the first move, it is abandoned and a gradient move is taken.

It can be observed that this technique is really an attempt to approximate the average gradient and not actually an abandonment of the gradient steep descent.[*]

In the problem of the shock isolator if $\ddot{X}_m$ is taken as the merit function, the merit curves for some pulse sets will exhibit a property we will call a "cusp". As was mentioned before, in different portions of the design space X_m will be the result of response maxima of different pulses. This shift will often be accomplished by an abrupt change in direction and magnitude of the gradient with, technically, the gradient being undefined at the cusp. This situation is illustrated in Fig. 10.

This is one case where the finite difference method for determining the gradient breaks down, (as, of course, do all methods because it doesn't exist). The gradient as computed usually points across the cusp and thus no move to improve the design can be made in its direction.

The situation is easy to detect because no move can be made and all progress stops. The solution to the difficulty is somewhat similar to that of the zigzag. Once progress stops a point on the cusp has been encountered; if a neighboring point on the cusp can be found, then these two points can be used to give a line running, at least for a distance, along the cusp " groove ".

This second point is easy to get since all gradients near this kind of a cusp are directed toward the cusp "groove". So several points about the first cusp point are tried in a more or less random fashion until one which is merely in the acceptable region is found. Gradient moves are then made from this point which usually lead back to the "groove" and a cessation of all progress. The vector between these two points is then used for the next move. The sense of the vector is obtained by considering which of the two cusp points has the better merit. The procedure is shown pictorially in Fig. 11.

This method is again an approximation to what might be thought of intuitively as the gradient of the cusp (if this offends the mathematical sensibilities, it is at least a vector in the direction we want to go).

---

[*] A slightly different form of this method has been used previously by Fedder[5].

The previously mentioned difficulty of moving along the constant merit curve was handled in the following manner. Since the normal (gradient direction) to the curve can be computed, its components may be used to give the tangent to the curve. In a typical situation one direction along this tangent leads into the unacceptable region and the other away from it (Fig. 12). If the curve were very flat then a move in the latter direction would be a move along the constant merit curve. For most cases, however, the curve is not so well behaved. If the curve is concave (Fig. 13), such a move will usually lead to a point of better merit than the bound point and there is no reason to attempt to find the curve of the bound point. After such a move gradient moves can be resumed until a constraint is again encountered.

If the curve is convex, (Fig. 14 a,b,c) the tangent move will lead to a point of worse merit than the bound point. In order to find the merit curve, small moves perpendicular to the tangent can be made until one of three things occur. 1) A constraint is encountered before the merit curve is passed through (Fig. 14a). This calls for shortening the tangent move and trying again. 2) Moves in the perpendicular direction show a reduced merit (Fig. 14b). This also calls for shortening the tangent and trying again. 3) The merit curve of the bound point is passed through or hit exactly (very rare). This is considered a success and gradient moves are made from this point (Fig. 14 c).

It is easy to see that this process will seldom result in a move which is truly along the merit curve. If the steps perpendicular to the tangent are quite small the point will be close to, but seldom on the curve. On the other hand, there seems to be no strong reason to ask for such precision; what really is accomplished is a move which gets "inside" the convex merit curve. The complete flow diagram and computer listing for the method described above is given in ref. 6.

EXAMPLE SYNTHESES

This section presents several illustrative cases of the operation of the synthesis program described above. These are presented graphically as synthesis paths superimposed on the family of merit curves and constraints.

The example problems presented are of two types; Find the spring stiffness k and damping coefficient c such that the system will isolate a mass m from a set of base induced acceleration shocks in the form of square pulses having a magnitude B_i and a duration T_i, i = 1,2,...n. The shock protection is to be such that

Type 1 problem: the maximum of the absolute acceleration maxima is to be minimized while having the maximum of the magnitudes of the relative displacement maxima less than a certain value.

Type 2 problem: the maximum of the magnitudes of the relative displacement maxima is to be minimized while having the maximum of the absolute acceleration maxima less than a certain value.

In addition there may in some cases be limits on the range of values which k and c may take.

Table I gives the case designation and figure number showing the synthesis path, the identification of the pulse set involved, the constraints placed upon the problem, and the initial and final values of k, c and the merit.

Table II lists the magnitude and duration of each pulse for the pulse sets used in these examples.

In viewing Figs. 15, 16 and 18-25 it should be borne in mind that the merit curves and behavior constraint curves are shown only for the purpose of demonstrating the modes of operation of the synthesis program. In practice they would not be known; indeed, if they were easily obtainable, the problem would reduce to one of plotting and the selection of the optimum could be done by inspection.

The curves could be obtained only by "gridding" the design space. This required on the average about ten times as much computer time as did the actual synthesis paths. In addition, the grids were run after the synthesis had been completed so that the pertinent region of the space was already known.

In the illustrations which follow, gradient moves which should be normal to the merit curves, will not appear to be so. This is due to the severe distortion of the scales which was necessary for clarity.

15.

Type 1 Problem, Minimize Acceleration (m = 1 lb-sec²/in)

Case	Fig. No.	Pulse Set	Bounds	Max Z in.	Initial Design K lb/in.	C lb-sec/in.	Merit in/sec²	Final Design K lb/in	C lb-sec/in.	Merit in/sec²
A	15	I	---	0.6	1000.0	3.0	580	409.9	15.9	346.4
B	16	I	---	0.6	200.0	300.0	1904	395.3	16.35	346.8
C	18	II	$0 \le c \le 100$	1.1	1000.0	24.0	1165	36.5	42.6	893.5
D	19	II	$0 \le c \le 100$	1.1	1000.0	39.0	1159	48.3	41.9	893.6
E	20	II	$0 \le c \le 30$	1.1	1000.0	29.0	1167	342.8	30.0*	932.4
F	21	II	---	0.6	1000.0	100.0	1317	341.2	69.8	1026.7
G	22	II	---	0.6	3000.0	12.0	1710	355.7	69.6	1027.4
H	23 A	II	$650 \le k$	1.1	1000.0	24.0	1165	650.4*	20.7	1008.1
J	23 B	II	$650 \le k$	1.1	1000.0	39.0	1159	650.8*	71.9	1066.8

Type 2 Problem, Minimize Displacement (m = 1 lb-sec²/in)

Case	Fig. No.		Bounds	Max Ẍ in/sec²	K lb/in.	C lb-sec/in	Merit in.	K lb/in	C lb-sec/in.	Merit in.
K	24	II	$0 \le c \le 30$	932.4	400.0	7.0	1.9	337	29.9*	1.1
L	25	II	$0 \le c \le 30$	932.4	50.0	0.0	7.0	333	29.9*	1.1

* Active Side Constraint

TABLE I
SUMMARY OF RESULTS

Set I		
B_1 = 2000 in/sec^2		T_1 = 0.001 sec.
B_2 = 200 in/sec^2		T_2 = 0.01 sec.
B_3 = 2000 in/sec^2		T_3 = 0.01 sec.
Set II		
B_1 = 2000 in/sec^2		T_1 = 0.01 sec.
B_2 = 1000 in/sec^2		T_2 = 0.05 sec.

TABLE II
PULSE SETS

Pulse Set I contains a dominant pulse (B_3,T_3) which rules the behavior of the system in the entire region of interest. In other words the syntheses depicted in Figs. 15 and 16 would be unaffected by the omission of pulses 1 and 2.

Figures 15 and 16 show two paths, each from a different starting design, for the solution of the same problem. In Fig. 15 the zigzag effect is quite in evidence. The moves from about k = 990 to 810, from 800 to 640, 620 to 580 and from 560 to 510 are all gradient approximate moves resulting from the zigzag feature of the program.

The designs indicated by small circles in the drawings are points from which gradient moves were taken; other points were checked for merit and/or constraints in the course of the synthesis but the points circled required about ten times as much computational time because of the finite difference gradient calculation, as did the points merely checked.

The synthesis path shown in Fig. 16 does not encounter the zigzag simply because the initial design and subsequent path do not cause it to pass through the region of rapidly changing gradient.

The solution for the problem shown in Figs. 15 and 16 is a true tangency between the acceleration merit curves and the displacement constraint. As will be seen in later examples, the imposition of side constraints can change this situation.

Figures 18 through 23 deal with pulse Set II in which there is not a dominant pulse which can be said a priori to control the design.

The wave like form of the merit curves in the subsequent figures may seem surprising---for a spring stiffness of about 700, for example, the addition of damping up to about 15 improves the design, further damping makes the design worse up to about 55 and then damping improves the design again up to 72. At this point the addition of damping suddenly causes the design to deteriorate and does so from C = 72 on up. The reason for this chain of events can be seen in Fig. 17 in which are plotted the actual acceleration response curves of the system for various values of damping.

The optimum for the problem shown in Figs. 18 and 19 is again a true tangency. The path for Fig. 18 involves a few zigzags followed by an extraordinarily long approximate move. This long approximate move will occur when the R happens to be just right.

The synthesis path shown in Fig. 19 involves the cusp move. The "groove" in this problem is fairly straight and therefore the first cusp direction suffices to move the design to a region where there is no further difficulty with the cusp.

Figure 20 shows a problem which is similar to the one shown in Figs. 18 and 19 but whereas the latter had no active side constraint the former does. By moving the upper bound on c from 100 to 30 the solution has changed from a true tangency point to a point of intersection between constraints. Since the acceptable region is so restricted in this problem a second path is not shown.

Figures 21 and 22 show the problem with a more severe restriction on displacement and no active side constraints. This change causes the solution to lie at a cusp point. This means that the pulses jointly control the optimum design.

It was observed in the above figures that the problem would have a relative minimum if a lower bound were placed on the spring stiffness above K = 400. This situation is shown in Fig. 23. The true optimum occurs at K = 650.0, C = 20.7 with a merit of 1008.1 and is due to path A. Path B yields a relative minimum of 1066.8.

This somewhat **artificially** induced relative minimum problem can be solved (i.e., the absolute minimum found) by using multiple paths-- the degree of confidence increasing with the number of paths which lead to the same minimum.

Figures 24 and 25 are the reverse, so to speak, of the problem shown in Fig. 20. In these the maximum relative displacement is to be minimized and a constraint is placed on the acceleration (type 2 problem). The acceleration limit chosen for Figs. 24 and 25 is the value of the minimum found in Fig. 20.

As expected, the minimum relative displacement found by the syntheses shown in Figs. 24 and 25 is the same as the upper limit placed on the displacement for Fig. 20.

There is nothing really profound in this result except that due to the relative simplicity of displacement as a merit function it provides a quick method of making "confidence checks" after the original problem has been run once.

CONCLUSIONS

This work has successfully demonstrated the feasibility of applying the synthesis concept to a problem in which the technology is dynamics. At the outset it was clear that if merit and constraint "rules" could be identified, problems in dynamics could be dealt with from the design parameter space viewpoint. However, there were two general areas which required investigation: (1) The practical question of explicit expression of the salient features of a dynamic analysis needed **exploration.** (2) The properties of merit and constraint functions required by the present ideas of synthesis were in question. In other words, it was not known if these functions were single valued, continuous, and without regions of zero gradient.

The investigation of these questions has lead to the development of a capability to optimize a two parameter dynamics problem. Specifically, the problem solved is:

Given a single-degree-of-freedom, spring-damper mass system to the base of which are applied n square pulses (each applied with the system at rest) of different magnitudes and/or duration, find the spring-damper combination which, within upper and lower bounds on stiffener and damping:

1) Causes the mass to experience the least maximum absolute acceleration due to any pulse while providing a maximum relative displacement between the mass and the base less than a prescribed value,

or

    2) causes the least maximum relative displacement while providing a maximum absolute acceleration of the mass less than a prescribed value.

The program requires as inputs:

    mass

    the value of the behavior constraint (i.e., displacement or acceleration for problems 1 and 2, respectively)

    an initial acceptable design

    three move increment sizes

    the constraint tolerance

    the zigzag angle criterion

    the tolerance for the gradient routine

    the pulse data

    design parameter bounds.

Due to the variety of "pathological" features which this problem presented, it is felt that the capability developed to cope with them will serve to optimize a very large class of general two dimensional problems. The development of techniques for handling these unusual features has been an important result of this work.

The feature of a completely bounded acceptable region (as shown in Figs. 24 and 25) has not always been expected. Due to the low dimensionality of the problem, it was not necessary to deal with this in a formal way, however, its exposure has lead to some new thoughts on the question of finding an initial design. One such thought is the idea of temporarily defining the constraining behavior function as a pseudo merit function. Then choosing an initial design which satisfies the design parameter bounds, running the synthesis program until this pseudo merit function is below the constraining value. The resulting design will then lie inside the bounded acceptable region.

The other unusual features were anomalies of the gradient field of the merit function. The rapidly changing but continuous gradient field which causes the zigzagging of gradient moves was dealt with in a manner which attempts, in a sense, to move along the center line of the "trough" of the function. The discontinuous gradient field caused by the changing of dominance from one pulse to another was dealt with by a similar method.

The recognition and solution of these situations, when they occur, has led to the development of methods which involve an elementary sort of learning process. The program is provided with the capacity to remember part of the history of the design path in which it is currently involved. It then evaluates this "experience" and makes decisions based upon its evaluation. Admittedly this memory is quite short and the decision process is fairly rudimentary. Any experience gained in one problem is currently carried over to the next only by the operator of the program. The idea of providing the capacity for gaining experience in synthesis problems opens a wide area of investigation in which automated redesign decisions can be made with the benefit of more information than just that available at the current design.

The scope of the current capability can be broadened by including the analysis of more types of pulses. In fact, a set of arbitrary pulses could be handled by providing a routine for **numerically integrating the** Duhamel integral and selecting the maxima from the resulting response. This would require a much longer analysis time than does the current method but would be accompanied by an attendant increment in generality.

Short of such generality, routines can be constructed which could analyze pulses such as semi-sine, ramp, over-pressure, etc. The synthesis could then handle a set of applied shocks which contained a variety of forms.

Another direction in which generality can be gained is in an increase in dimensionality. The single degree of freedom systems may have more than two design parameters. For example, the spring may be nonlinear of the form $F = k_1 x + k_2 x |x|$ or be piecewise linear as with a snubbing device. The damper may also be nonlinear requiring more than one parameter to characterize its action.

APPENDIX A

The differential equation of the system shown in Figure 1 is:

$$m\ddot{x} = k(y - x) + c(\dot{y} - \dot{x}) \tag{A1}$$

where for the square pulses considered:

$$\ddot{y}_i(t) = B_i \qquad 0 < t \le T_i$$

$$i = 1, 2, \ldots N$$

$$= 0 \qquad T_i < t < +\infty \tag{A2}$$

Taking:

$$y - x = z, \qquad \frac{k}{m} = \omega_o^2, \qquad \frac{c}{m} = 2n$$

then:

$$\ddot{z}_i + 2n\dot{z}_i + \omega_o^2 z_i = \ddot{y}_i(t) \tag{A3}$$

The desired quantities are:

$$z_{m_i} = \max |z_i(t)| \qquad i = 1, 2, \ldots N$$

$$\ddot{x}_{m_i} = \max |\ddot{x}_i(t)| \qquad i = 1, 2, \ldots N$$

In what follows the subscript i will be dropped. It should be kept in mind however that for synthesis purposes what is ultimately needed are the values of $\ddot{X}_m$ and Z_m.

$$Z_m = \max \left[\max |z_1(t)|, \ldots, \max |z_n(t)| \right]$$

$$\ddot{X}_m = \max \left[\max |\ddot{x}_1(t)|, \ldots \max |\ddot{x}_n(t)| \right]$$

The solution of (A3) depends upon the relative values of ω_o and n.

<center>Subcritical Damping $(n < \omega_o)$, $t \leq T$</center>

$$z(t) = e^{-nt}(C_1 \cos \omega_A t + C_2 \sin \omega_A t) + \frac{B}{\omega_o^2} \qquad (A4)$$

where

$$\omega_A^2 = \omega_o^2 - n^2 \qquad (A5)$$

C_1 and C_2 may be determined from:

$$z(0) = 0$$

$$\dot{z}(0) = 0$$

which results in:

$$z(t) = -\frac{B e^{-nt}}{\omega_o^2} \left(\cos \omega_A t + \frac{n}{\omega_A} \sin \omega_A t\right) + \frac{B}{\omega_o^2} \qquad (A6)$$

$$\dot{z}(t) = \frac{B \omega_A}{\omega_o^2} e^{-nt} \left(1 + \frac{n^2}{\omega_A^2}\right) \sin \omega_A t \qquad (A7)$$

$$\ddot{z}(t) = \frac{B \omega_A}{\omega_o^2} e^{-nt} \left(1 + \frac{n^2}{\omega_A^2}\right)(\omega_A \cos \omega_A t - n \sin \omega_A t) \qquad (A8)$$

For the maxima of z:

$$\dot{z}(t_o) = 0$$

$$\frac{B \omega_A}{\omega_o^2} e^{-nt_o} \left(1 + \frac{n^2}{\omega_A^2}\right) \sin \omega_A t_o = 0 \qquad (A9)$$

$$\sin \omega_A t_o = 0$$

$$t_o = 0, \frac{\pi}{\omega_A}, \cdots \frac{N\pi}{\omega_A} \leq T \qquad (A10)$$

where T is the duration of the pulse.

The relation (A10) indicates:

$$\omega_A \geq \frac{\pi}{T}$$

in order that a maximum may occur during the pulse.

Substituting (A10) into (A6)

$$z(t_o) = \frac{B}{\omega_o^2} \left[1 + (-1)^{N+1} e^{-\frac{n\pi N}{\omega_A}} \right] \tag{A11}$$

which is greatest for N = 1

$$Z_m = \frac{B}{\omega_o^2} \left(1 + e^{-\frac{n\pi}{\omega_A}} \right) \tag{A12}$$

$$t \leq T_i, \qquad \omega_A \geq \frac{\pi}{T}$$

For the maxima of $\ddot{x}(t)$, $t \leq T$; from (A8) and the fact that:

$$\ddot{x}(t) = \ddot{y}(t) - \ddot{z}(t)$$

$$\ddot{x} = B - \frac{B\omega_A}{\omega_o} \left[\left(1 + \frac{n^2}{\omega_A^2} \right) \sin(\omega_A t + \varphi) \right] e^{-nt} \tag{A13}$$

where $\varphi = -\tan^{-1} \frac{\omega_A}{n}$.

$$\dddot{x}(t_o) = 0$$

$$\dddot{x}(t) = -\frac{B\omega_A}{\omega_o} \left(1 + \frac{n^2}{\omega_A^2} \right) e^{-nt} \left[\omega_A \cos(\omega_A t + \varphi) - n \sin(\omega_A t + \varphi) \right] \tag{A14}$$

24

$$0 = \omega_A \cos(\omega_A t_0 + \varphi) - n \sin(\omega_A t_0 + \varphi)$$

$$t_0 = \frac{2 \tan^{-1}(\frac{\omega_A}{n})}{\omega_A} \leq T \tag{A15}$$

Substituting into (A13)

$$\ddot{x}(t_0) = B - \frac{B \omega_A}{\omega_0}(1 + \frac{n^2}{\omega_A^2}) \sin(2 \tan^{-1}\frac{\omega_A}{n} - \tan^{-1}\frac{\omega_A}{n}) e^{-nt_0} \tag{A16}$$

which reduces to

$$\ddot{X}_m = B\left[1 + e^{-nt_0}\right] \quad t_0 \leq T \tag{A17}$$

However, the maximum may occur at $t = T$ in cases where $\ddot{x}(T) \neq 0$ because $\ddot{x}(t)$ is discontinuous at T (see Fig. 17). This is not true of $z(t)$ because $\dot{z}(t)$ is continuous at T.

Subcritical Damping $t > T$

$$z(t') = e^{-nt'}(K_1 \cos \omega_A t' + K_2 \sin \omega_A t') \tag{A18}$$

where $t = t' + T$.

K_1 and K_2 may be determined from:

$$z(T) = -\frac{B}{\omega_0^2} e^{-nT}(\cos \omega_A T + \frac{n}{\omega_A} \sin \omega_A T) + \frac{B}{\omega_0^2} \tag{A19}$$

$$\dot{z}(T) = \frac{B \omega_A}{\omega_0^2} e^{-nT}(1 + \frac{n^2}{\omega_0^2}) \sin \omega_A T \tag{A20}$$

which come from (A6), (A7).

This results in:

$$K_1 = z(T) \tag{A21}$$

$$K_2 = \frac{\dot{z}(T) + nz(T)}{\omega_A} \tag{A22}$$

Adopting the notation

$$G = 2 n\dot{z}(T) + \omega_0^2 z(T) \tag{A23}$$

$$Q = \frac{(2n^2 - \omega_0^2) \dot{z}(T) + n \omega_0^2 z(T)}{\omega_A} \tag{A24}$$

and dropping the primes on t (i.e., considering this a new problem with new initial conditions)

$$z(t) = e^{-nt} \left[z(T) \cos \omega_A t + (\frac{\dot{z}(T) + nz(T)}{\omega_A}) \sin \omega_A t \right] \tag{A25}$$

$$\dot{z}(t) = e^{-nt} \left[\dot{z}(T) \cos \omega_A t - (\frac{n \dot{z}(T) + \omega_0^2 z(T)}{\omega_A}) \sin \omega_A t \right] \tag{A26}$$

$$\ddot{z}(t) = -\ddot{x}(t) = e^{-nt} (Q \sin \omega_A t - G \cos \omega_A T) \tag{A27}$$

For the maxima of z, $t > T$:

$$\dot{z}(t_0) = 0$$

$$0 = \dot{z}(T) \cos \omega_A t_0 - (\frac{n\dot{z}(T) + \omega_0^2 z(T)}{\omega_A}) \sin \omega_A t_0 \tag{A28}$$

$$t_0 = \frac{1}{\omega_A} \tan^{-1} \left[\frac{\dot{z}(T) \omega_A}{n\dot{z}(T) + \omega_0^2 z(T)} \right] \tag{A29}$$

$$Z_m = e^{-nt_o} \left[z(T) \cos \omega_A t_o + \left(\frac{\dot{z}(T) + nz(T)}{\omega_A} \right) \sin \omega_A t_o \right]$$

$$\text{(A30)}$$

$$\frac{\omega_A T}{\pi} \leq 1$$

When $1 \geq \frac{\pi}{\omega_A T}$ the maxima of z that occur during $0 < t \leq T$ will be greater than or equal to the maxima for $t > T$. If $\frac{\omega_A T}{\pi} \leq 1$ then the maximum occurs for $t > T$.

If $n = 0$ (A30) reduces to

$$Z_m = \frac{2B}{\omega_o^2} \sin \frac{\omega_o T}{2}, \quad \frac{\omega_A T}{\pi} \leq 1 \quad \text{(A31)}$$

For the maxima of $\ddot{x}$, $t > T$:

$$\dot{\ddot{x}}(t_o) = 0$$

$$\dot{\ddot{x}}(t) = -e^{-nt} \left[(\omega_A Q + nG) \cos \omega_A t + (\omega_A G - nQ) \sin \omega_A t \right]$$

$$\text{(A32)}$$

$$0 = (\omega_A Q + nG) \cos \omega_A t_o - (nQ - \omega_A G) \sin \omega_A t_o \quad \text{(A33)}$$

$$t_o = \frac{1}{\omega_A} \tan^{-1} \left[\frac{\omega_A Q + nG}{nQ - \omega_A G} \right] \quad \text{(A34)}$$

If t_o from equation (A34) is less than zero, its value must be adjusted by the addition of $\frac{\pi}{\omega_A}$.

$$\ddot{X}_m = - e^{-nt_o} \left[Q \sin \omega_A t_o - G \cos \omega_A t_o \right] \tag{A35}$$

$$\frac{\omega_A T}{\pi} \leq 1$$

The acceleration at the end of the pulse from equation (A13) is:

$$\ddot{X}(T) = B - \frac{B \omega_A}{\omega_o} \left[(1 + \frac{n^2}{\omega_A^2}) \sin (\omega_A T_i + \varphi) \right] e^{-nT} \tag{A36}$$

If $n = 0$:

$$\ddot{X}_m = 2B \sin \frac{\omega_o T}{2}, \qquad t > T \tag{A37}$$

$$\ddot{X}_m = 2B, \qquad t_o < T, \qquad \omega_o \geq \frac{\pi}{T} \tag{A38}$$

Critical Damping $(n = \omega_o)\ t \leq T$

$$z(t) = (\alpha + \beta t) e^{-nt} + \frac{B}{\omega_o^2} \tag{A39}$$

$$\dot{z}(t) = e^{-nt} (\beta - n\alpha - n\beta t) \tag{A40}$$

α and β may be determined from:

$$z(0) = 0$$
$$\dot{z}(0) = 0$$

28

$$\alpha \;=\; -\;\frac{B}{\omega_o{}^2} \;=\; -\;\frac{B}{n^2}$$

$$\beta \;=\; -\;\frac{B}{\omega_o} \;=\; -\;\frac{B}{n}$$

Therefore:

$$z(t) \;=\; \frac{B}{n^2}\left[1 - (1 + nt)\,e^{-nt}\right] \tag{A41}$$

$$\dot{z}(t) \;=\; Bt\;e^{-nt} \tag{A42}$$

Equation (A42) shows that no maximum of $z(t)$ may occur during $0 < t \leq T$. For the maxima of $\ddot{x}$ $t < T$

$$\dot{\ddot{x}}(t) \;=\; B - B\,e^{-nt}\,(1 - nt) \tag{A43}$$

$$\dot{\dddot{x}}(t) \;=\; -B\left[n^2 t\,e^{-nt} - 2\,n\,e^{-nt}\right] \tag{A44}$$

$$0 \;=\; n^2 t_o\,e^{-nt_o} \;-\; 2n\,e^{-nt_o} \tag{A45}$$

$$t_o \;=\; \frac{2}{n} \;\leq\; T \tag{A46}$$

$$\ddot{X}_m \;=\; B\left[1 - e^{-2}\,(1 - 2)\right] \;=\; 1.13534B \tag{A47}$$

$$\frac{2}{n} \;\leq\; T$$

Critical Damping $t > T$

$$z(t') \;=\; (\gamma + \delta\,t')\,e^{-nt'} \tag{A48}$$

γ and δ may be obtained from the initial conditions:

$$z(T) = \frac{B}{n^2} \left[1 - (1 + nT) e^{-nT} \right] \tag{A49}$$

$$\dot{z}(T) = BT e^{-nT} \tag{A50}$$

$$\gamma = z(T) \tag{A51}$$

$$\delta = \dot{z}(T) = nz(T) \tag{A52}$$

From which (dropping the primes on t):

$$z(t) = \left\{ z(T) + \left[z(T) + nz(T) \right] t \right\} e^{-nt} \tag{A53}$$

$$\dot{z}(t) = \left\{ \dot{z}(T) - \left[n\dot{z}(T) + n^2 z(T) \right] t \right\} e^{-nt} \tag{A54}$$

$$\ddot{z}(t) = -\ddot{X}(t) = \left\{ \left[n^2 \dot{z}(T) + n \, z(T) \right] t - \left[2n \, \dot{z}(T) + n^2 x(T) \right] \right\} e^{-nt} \tag{A55}$$

For the maximum z :

$$0 = \dot{z}(T) - \left[n \dot{z}(T) + n^2 z(T) \right] t_o$$

$$t_o = \frac{\dot{z}(T)}{n\dot{z}(T) + n^2 Z(T)} \tag{A56}$$

$$Z_m = \frac{B}{n^2} (1 - e^{-nT}) \, e^{-nt_o} \tag{A57}$$

30

For the maximum of $\overset{..}{x}$:

$$\overset{...}{x} = -e^{-nt}\left\{\left[2\,n^2\dot{z}(T) + n^3\dot{z}(T)\right] - \left[n^3\dot{z}(T) + n^4z(T)\right]t + \left[n^2\dot{z}(T) + n^3z(T)\right]\right\}$$

(A58)

$$0 = 3\dot{z}(T) + 2n\,z(T) - \left[n\,\dot{z}(T) + n^2z(T)\right]t_o$$

(A59)

$$t_o = \frac{3\dot{z}(T) + 2n\,z(T)}{n\,\dot{z}(T) + n^2z(T)}$$

(A60)

$$\overset{..}{X}_m = \left|\,e^{-nt_o}\left[n\,\dot{z}(T) + n^2z(T)\right]\,\right|$$

(A61)

The acceleration at the end of the pulse is:

$$\overset{..}{x}(T) = -\left\{\left[n^2\dot{z}(T) + n^3z(T)\right]T - \left[2n\,\dot{z}(T) + n^2z(T)\right]\right\}\,e^{-nT}$$

(A62)

Overdamped $(n > \omega_o)$, $0 < t \leq T$

$$z(t) = e^{-nt}\left(A\cos\omega_{od}\,t + D\sinh\omega_{od}\,t\right) + \frac{B}{\omega_o^2}$$

(A63)

where:

$$\omega_{od}^2 = n^2 - \omega_o^2$$

A and D may be determined from:

$$z(0) = \dot{z}(0) = 0$$

$$A = -\frac{B}{\omega_o^2}$$

$$D = -\frac{n\,B}{\omega_{od}\,\omega_o^2}$$

(A64)

Then

$$z(t) = \frac{B}{\omega_o^2} \left\{ 1 - e^{-nt} \left[\cosh \omega_{od} t + \frac{n}{\omega_{od}} \sinh \omega_{od} t \right] \right\} \quad \text{(A65)}$$

$$\dot{z}(t) = e^{-nt} \left[\left(\frac{n^2 B}{\omega_{od} \omega_o^2} - \frac{\omega_{od} B}{\omega_o^2} \right) \sinh \omega_{od} t \right] \quad \text{(A66)}$$

$$\ddot{z}(t) = \frac{B}{\omega_{od}} e^{-nt} \left[\omega_{od} \cosh \omega_{od} t - n \sinh \omega_{od} t \right] \quad \text{(A67)}$$

$$\ddot{x} = B - \ddot{z}(t) \quad \text{(A68)}$$

$$\dddot{x} = -\frac{B e^{-nt}}{\omega_{od}} \left[(n^2 + \omega_{od}^2) \sinh \omega_{od} t - 2n \omega_{od} \cosh \omega_{od} t \right] \quad \text{(A69)}$$

For the maximum of $z(t)$:

$$0 = \sinh \omega_{od} t_o$$

which indicates that $z(t)$ has no maximum $0 < t \leq T$.

For the maximum of $\ddot{x}(t)$:

$$0 = (n^2 + \omega_{od}^2) \sinh \omega_{od} t_o - 2n \omega_{od} \cosh \omega_{od} t_o \quad \text{(A70)}$$

$$\frac{2n \omega_{od}}{n^2 + \omega_{od}^2} = \tanh \omega_{od} t < 1 \quad \text{(A71)}$$

$$t_o = \frac{\tanh^{-1}}{\omega_{od}} \left(\frac{2n \omega_{od}}{n^2 + \omega_{od}^2} \right) \leq T \quad \text{(A72)}$$

If $\dfrac{2n \omega_{od}}{n^2 + \omega_{od}^2} \geq 1$ no maximum occurs for $t < T$.

32

The acceleration at the end of the pulse:

$$\ddot{X}(T) = B \left\{ 1 - \frac{e^{-nt}}{\omega_{od}} \left[\omega_{od} \cosh \omega_{od} T - n \sinh \omega_{od} T \right] \right\} \quad \text{(A73)}$$

Overdamped $t > T$

$$z(t') = e^{-nt'} (S \cosh \omega_{od} t' + R \sinh \omega_{od} t') \quad \text{(A74)}$$

For the initial condition:

$$z(T) = \frac{B}{\omega_o^2} \left\{ 1 - e^{-nT} \left[\cosh \omega_{od} T + \frac{n}{\omega_{od}} \sinh \omega_{od} T \right] \right\} \quad \text{(A75)}$$

$$\dot{z}(T) = e^{-nT} \left[\left(\frac{n^2 B}{\omega_{od} \omega_o^2} - \frac{\omega_{od} B}{\omega_o^2} \right) \sinh \omega_{od} T \right] \quad \text{(A76)}$$

$$S = z(T)$$

$$R = \frac{\dot{z}(T) + nz(T)}{\omega_{od}} \quad \text{(A78)}$$

from which (again dropping the primes on t):

$$z(t) = e^{-nt} \left[z(T) \cosh \omega_{od} t + \frac{\dot{z}(T) + nz(T)}{\omega_{od}} \sinh \omega_{od} t \right]$$
$$\text{(A79)}$$

$$\dot{z}(t) = e^{-nt} \left\{ - \left(\frac{\omega_o^2 z(T) + n\dot{z}(T)}{\omega_{od}} \right) \sinh \omega_{od} t + \dot{z}(T) \cosh \omega_{od} t \right\}$$
$$\text{(A80)}$$

$$\ddot{z}(t) = - \ddot{x}(t) = e^{-nt} \left[V \sinh \omega_{od} t - W \cosh \omega_{od} t \right] \quad \text{(A81)}$$

Where:

$$V = \frac{(2 n^2 - \omega_o^2) \dot{z}(T) + n \omega_o^2 z(T)}{\omega_{od}} \qquad (A82)$$

$$W = \omega_o^2 z(T) + 2 n \dot{z}(T) \qquad (A83)$$

For the maximum of z :

$$\dot{z}(t_o) = 0$$

$$\frac{\omega_{od} \dot{z}(T)}{\omega_o^2 z(T) + n\dot{z}(T)} = \tanh \omega_{od} t_o < 1 \qquad (A84)$$

$$t_o = \frac{1}{\omega_{od}} \tanh^{-1} \left[\frac{\omega_{od} \dot{z}(T)}{\omega_o^2 z(T) + n\dot{z}(T)} \right] \qquad (A85)$$

If $\dfrac{\omega_{od} \dot{z}(T)}{\omega_o^2 z(T) + n\dot{z}(T)} \geq 1$ there is no maximum for $t > T$.

For the maximum of $\ddot{x}$:

$$\frac{\omega_{od} V + nW}{nV + \omega_{od} W} = \tanh \omega_{od} t_o \qquad (A86)$$

$$t_o = \frac{1}{\omega_{od}} \tanh^{-1} \left[\frac{\omega_{od} V + nW}{nV + \omega_{od} W} \right] \qquad (A87)$$

REFERENCES

1. Schmit, L. A., and Kicher, T. P., "Structural Synthesis
    of Symmetric Waffle Plates with Integral Orthogonal
    Stiffeners," Engineering Design Center, EDC Report No.
    2-61-1, Case Institute of Technology, Oct., 1961.

2. Schmit, L. A., and Mallett, R. H., "Design Synthesis in
    Multidimensional Space with Automated Material Selection",
    Engineering Design Center, EDC Report No. 2-62-2, Case
    Institute of Technology, Aug., 1962.

3. Schmit, L. A., and Kicher, T. P., "Structural Synthesis of
    Symmetric Waffle Plate," NASA Technical Note D-1691,
    December, 1962.

4. Best, G., "A Method of Structural Weight Minimization
    Suitable for High-Speed Digital Computers," AIAA Journal,
    Vol. I, No. 2, Tech. Note, pp. 478-479, Feb., 1963.

5. Feder, D. P., "Automatic Lens Design Methods" Journal
    Amer. Opt. Soc., Vol. 47, No. 10, Oct., 1957.

6. Schmit, L. A., and Fox, R. L., "Synthesis of a Simple
    Shock Isolator", Engineering Design Center, EDC
    Report No. 2-63-4, Case Institute of Technology.

$$z = y - x$$
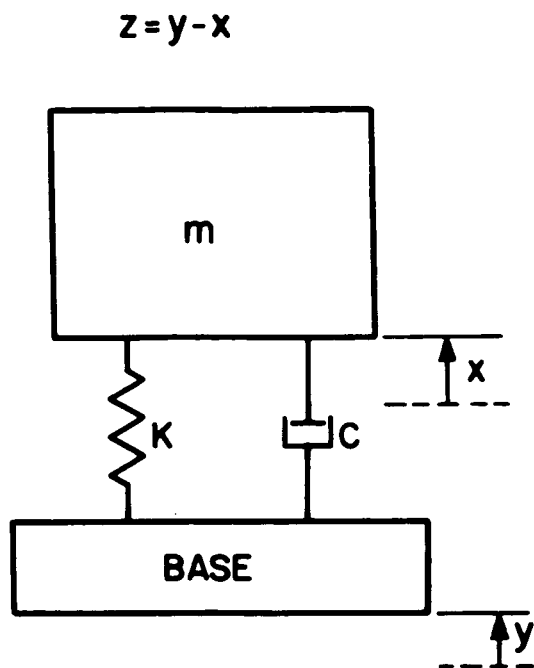
Figure 1. The Simple System
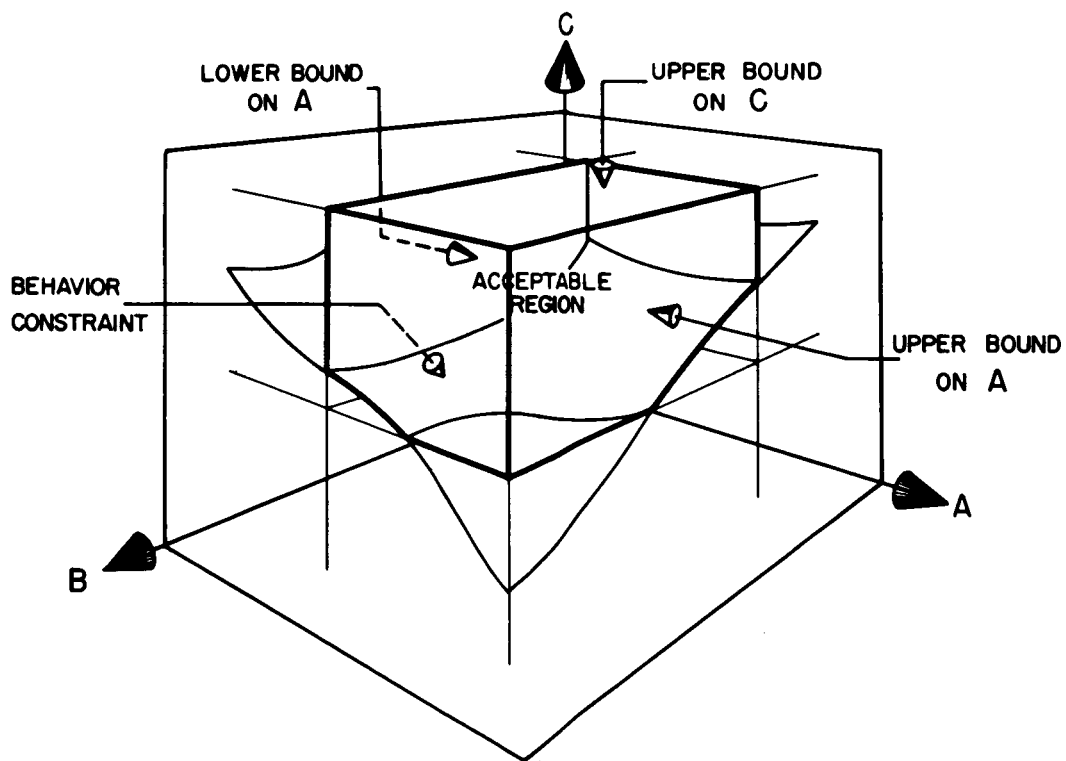
Figure 2. Design Parameter Space, N = 3.
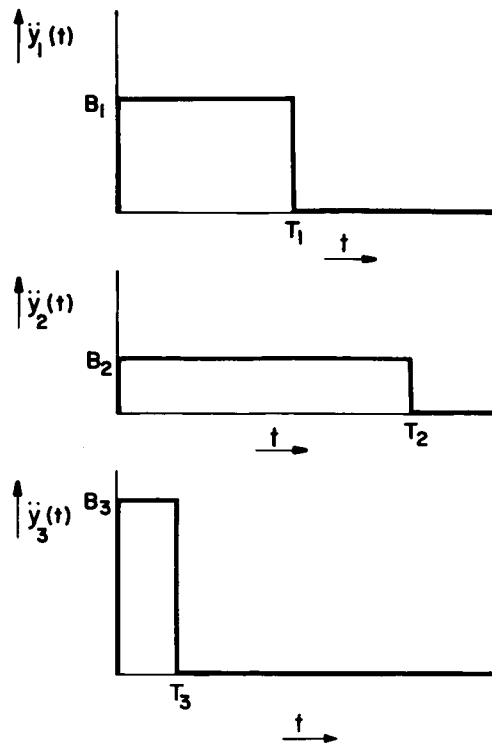
Figure 3. The Square Pulse
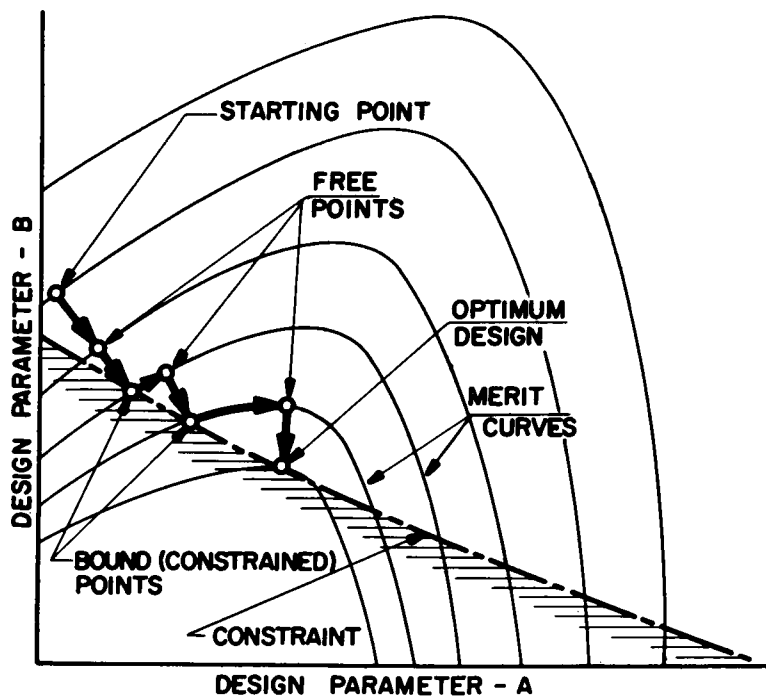
Figure 4. The Gradient Alternate Step Method.
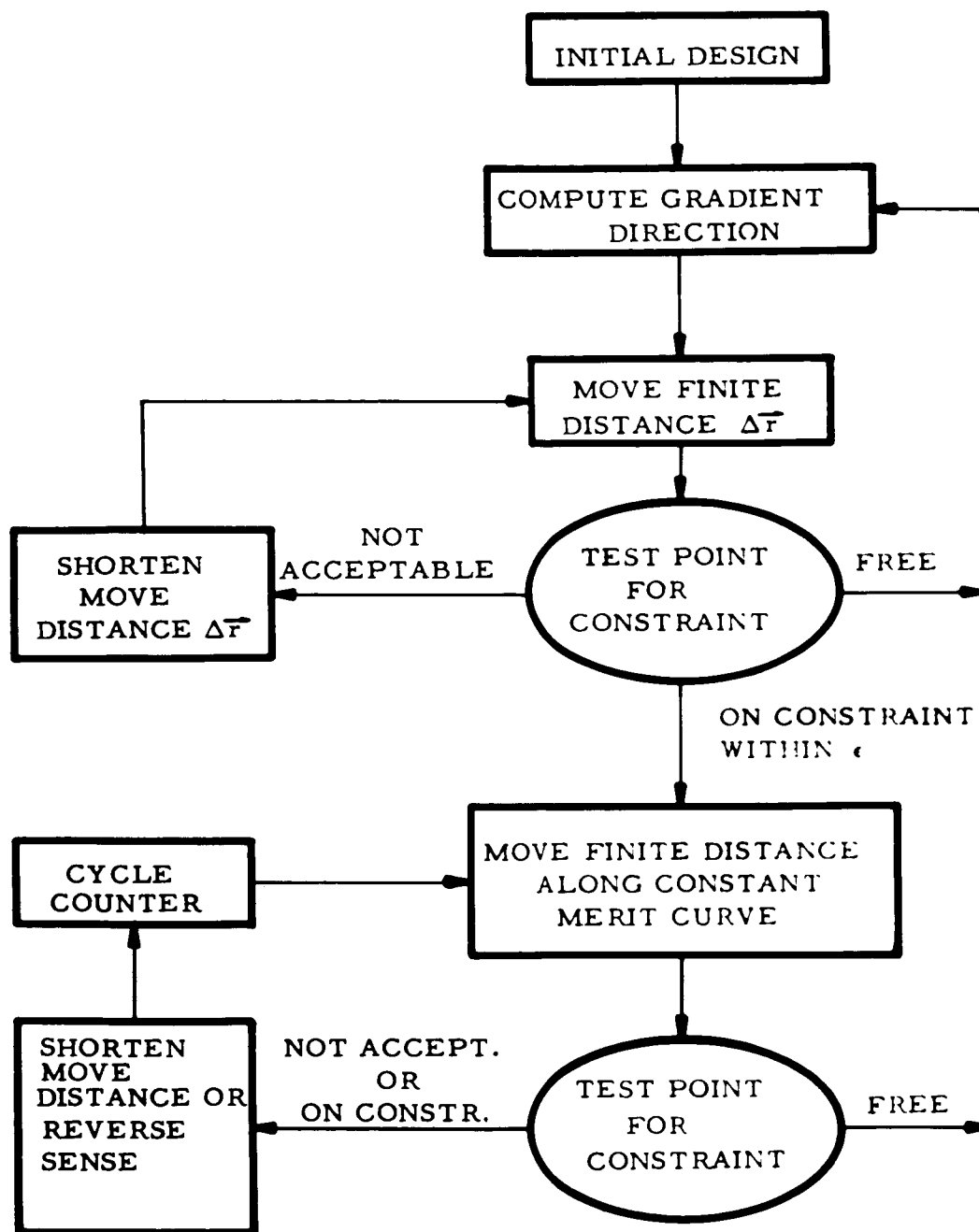
Figure 5. Basic Flow Diagram for Steep Descent - Alternate Step.
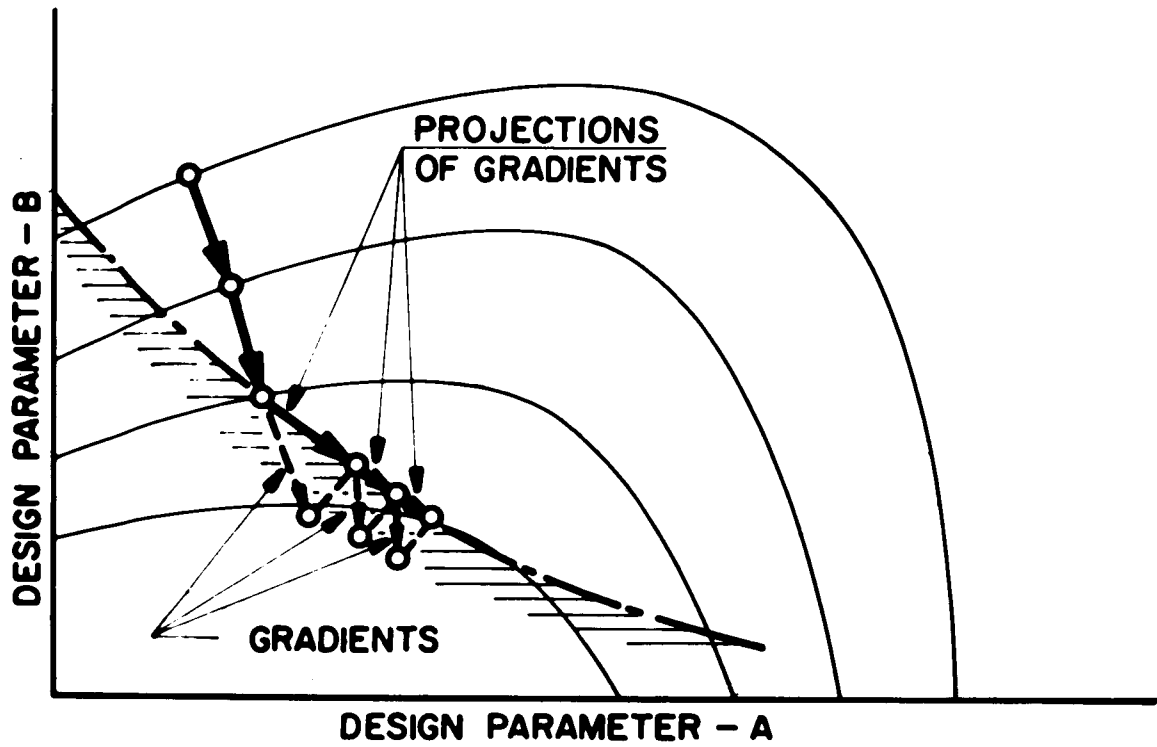
Figure 6. The Constrained Gradient Method.
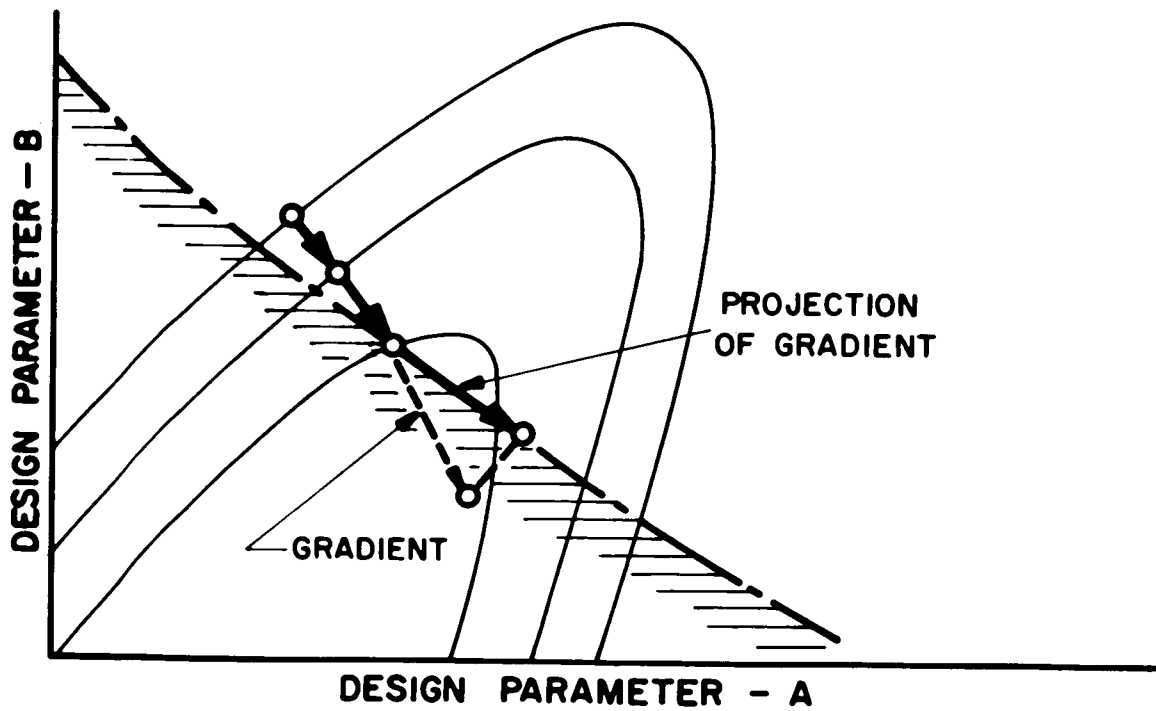
Figure 7. Constrained Gradient Method, Move too Long.
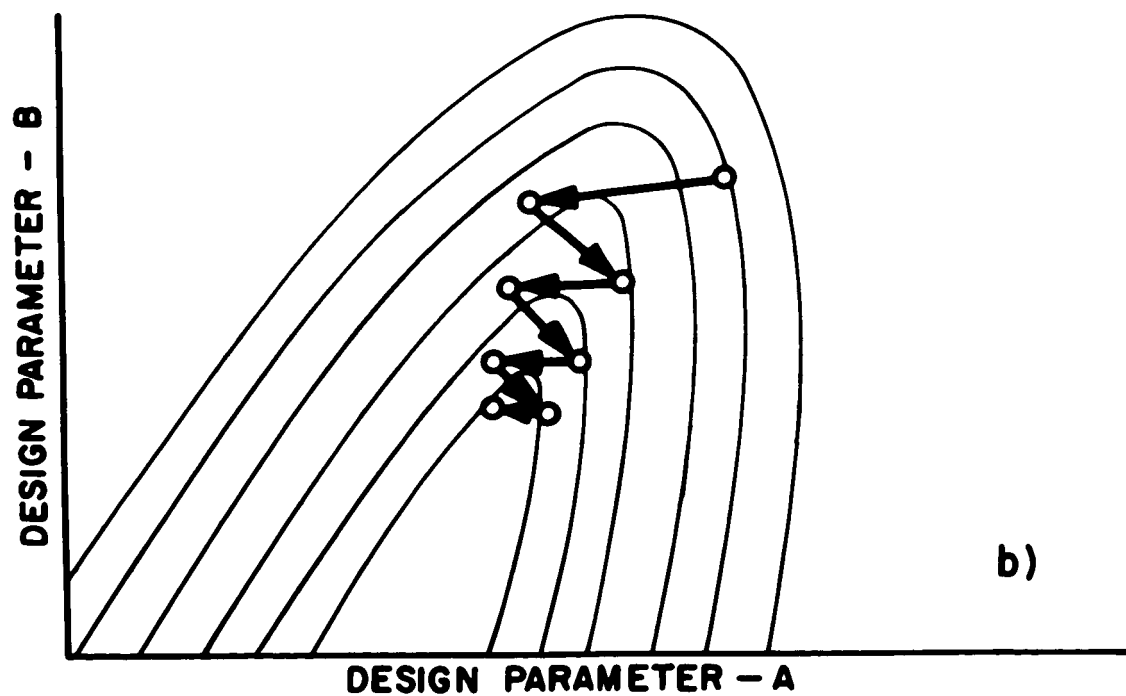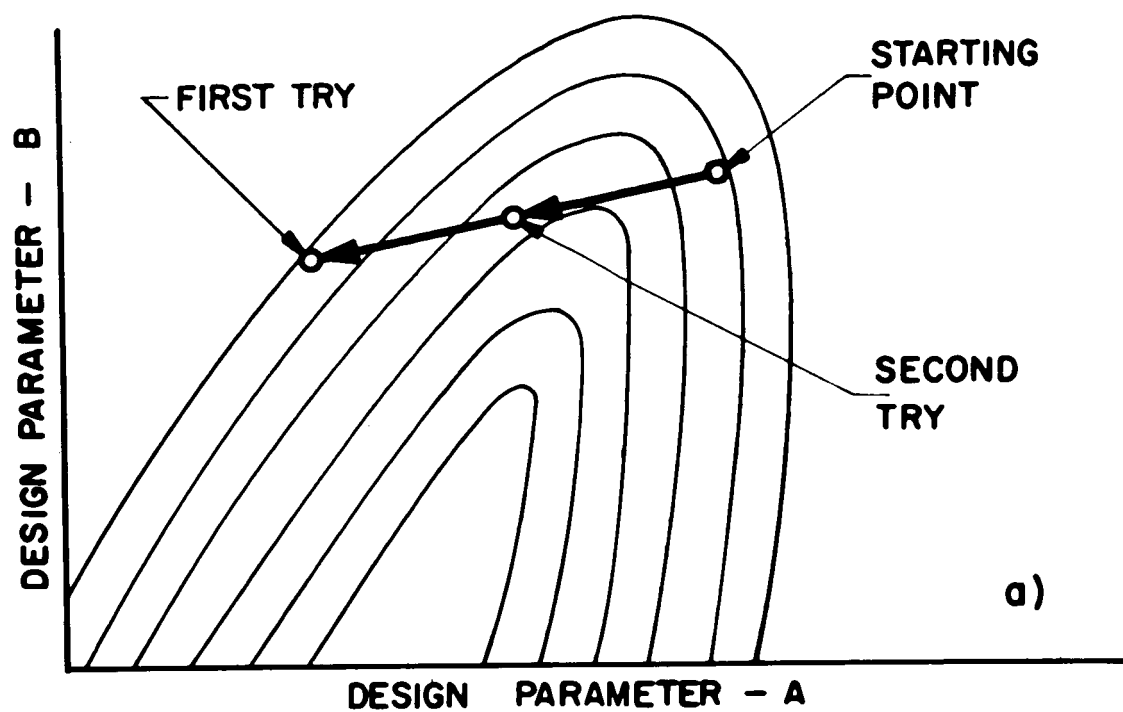
Figure 8. The Zig-Zag Situation.

DESIGN PARAMETER – B

DESIGN PARAMETER – A

THE VECTORS
IN THE ZIG–ZAG

$\vec{m_1}$

$\vec{m_2}$

$\vec{R}$

$\vec{V_3}$

$\vec{V_2}$

$\vec{V_1}$

DESIGN PARAMETER - B

DESIGN PARAMETER – A

THE ZIG–ZAG
GRADIENT
APPROXIMATION

$\vec{V_1}$

$\vec{R}$
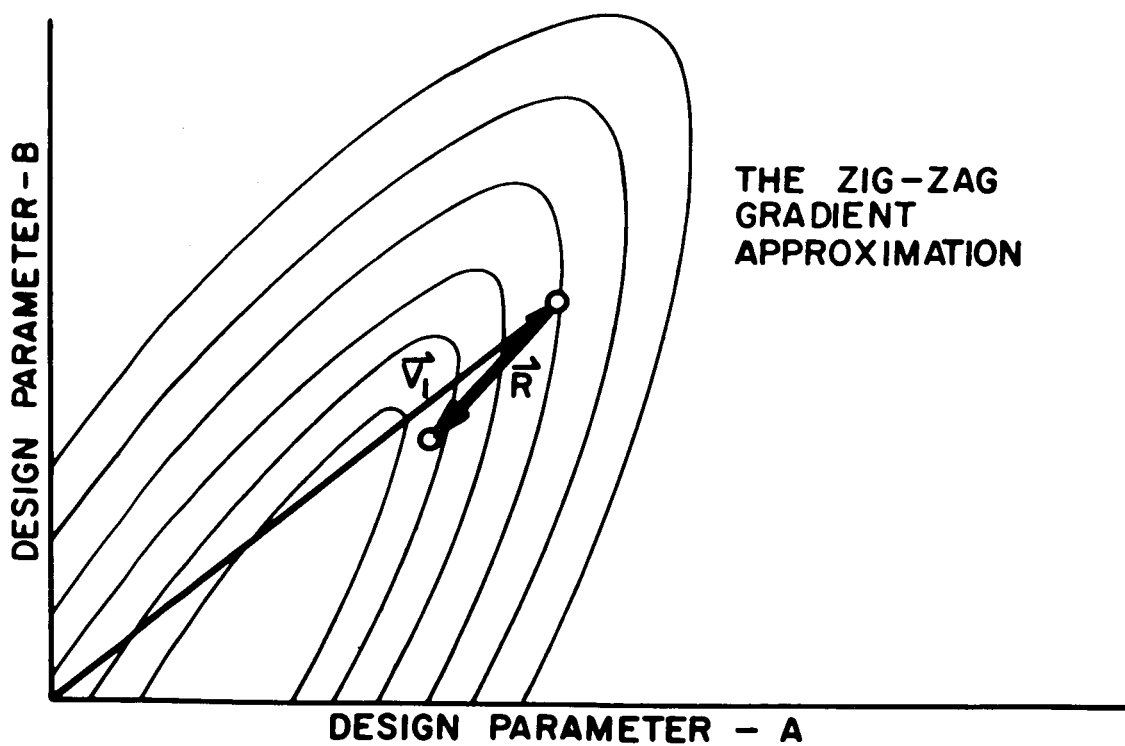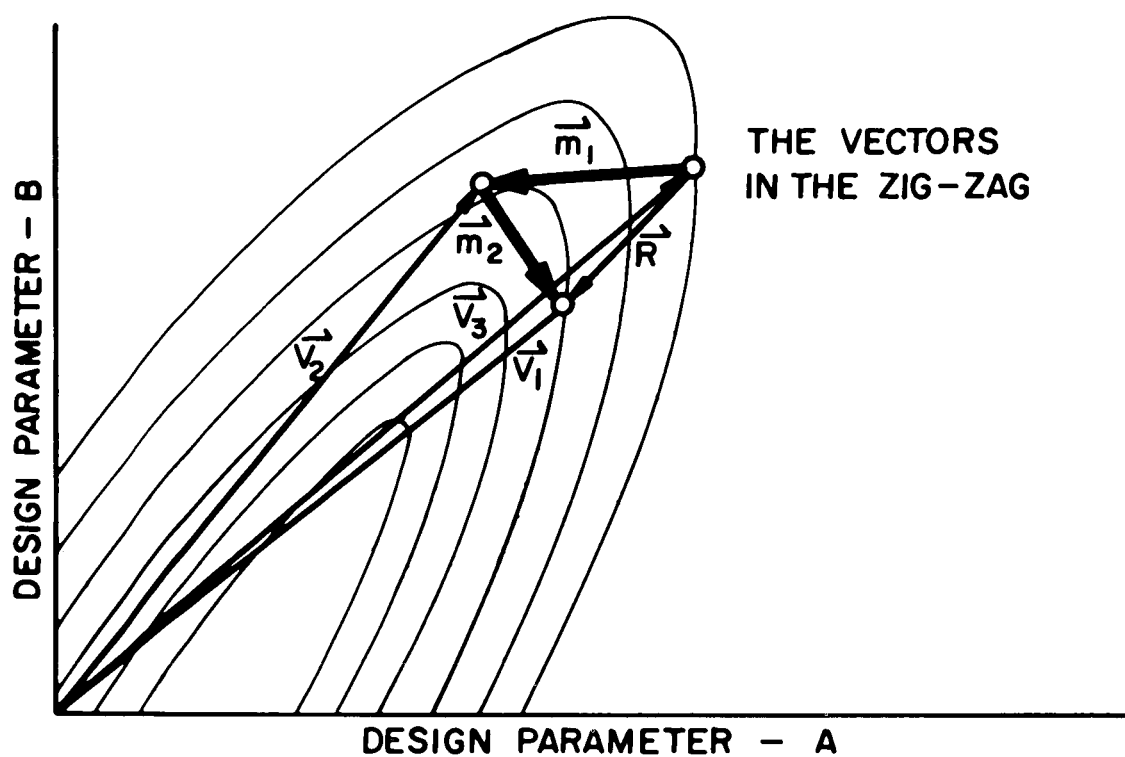
Figure 9. The Zig-Zag Vectors.
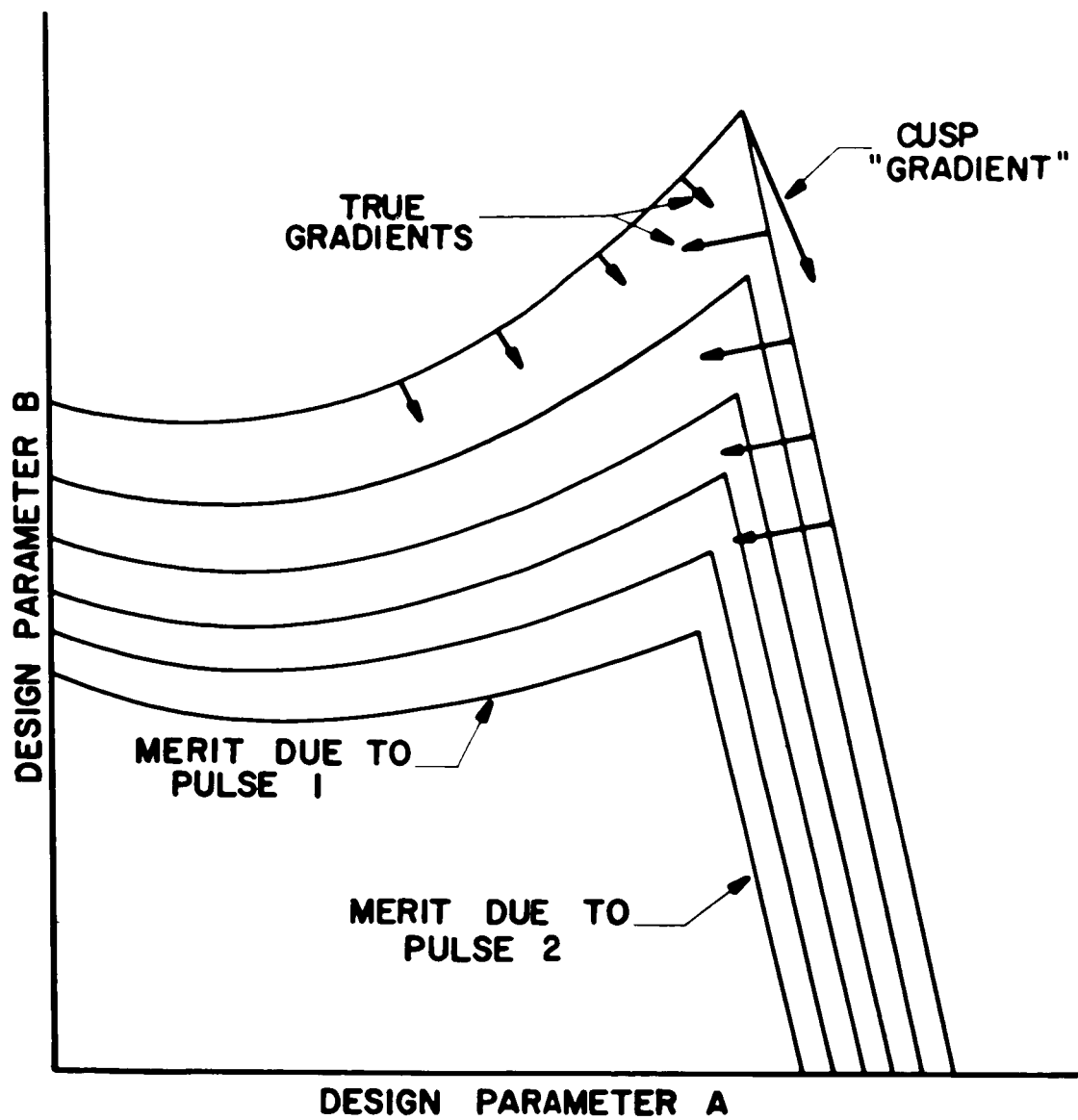
Figure 10. The Cusp.
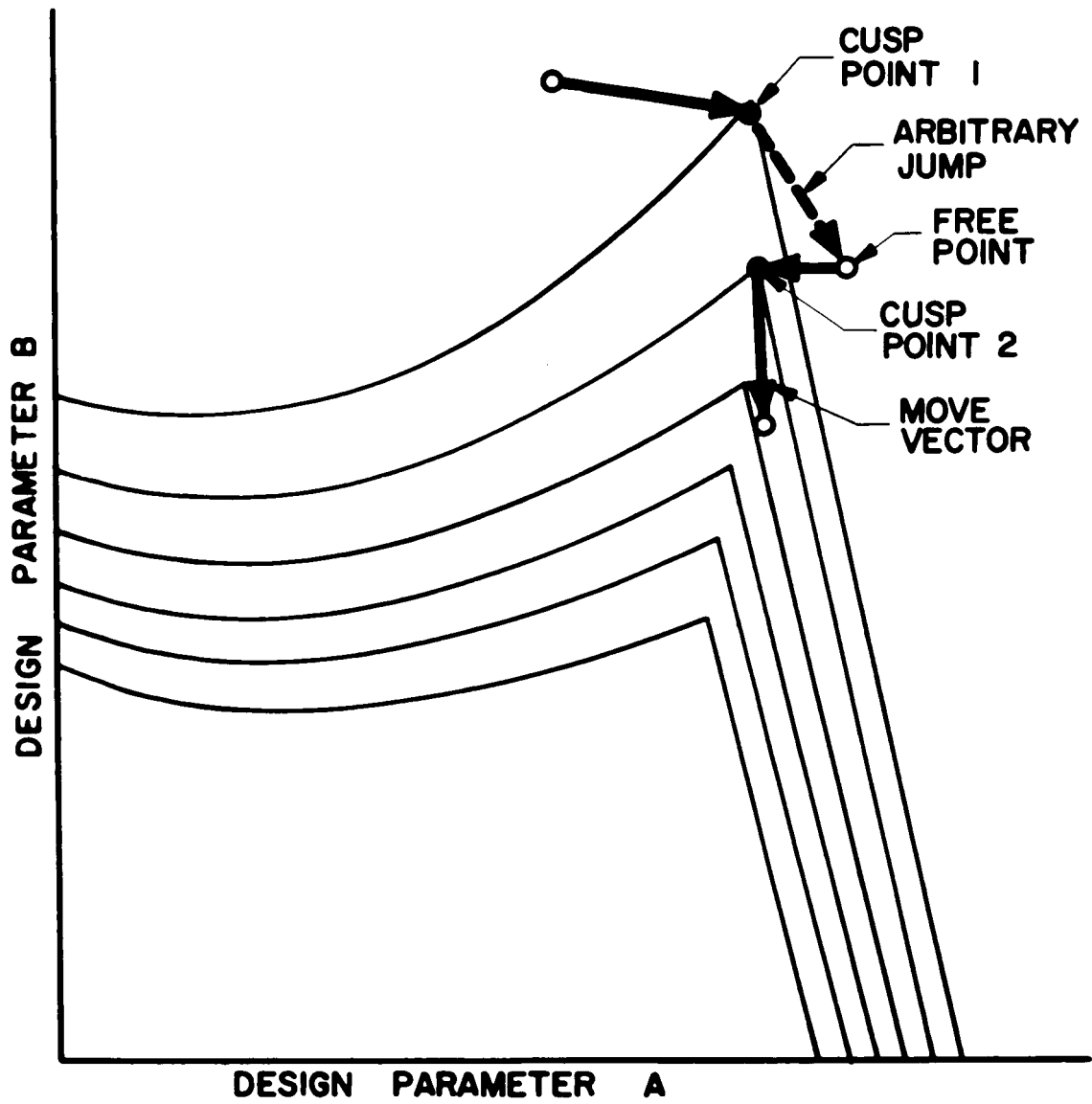
Figure 11. The Cusp Move.
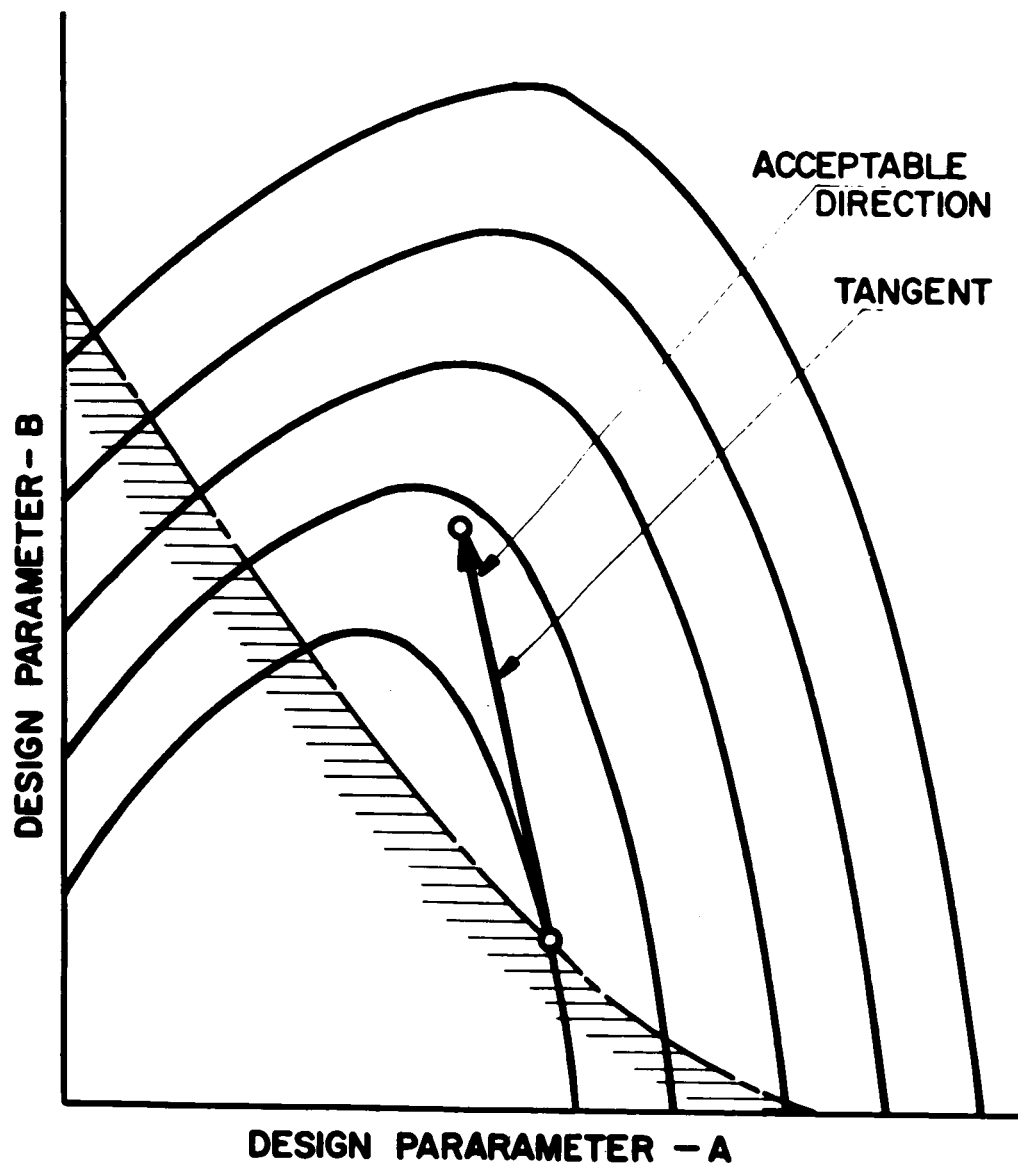
Figure 12. The Tangent.
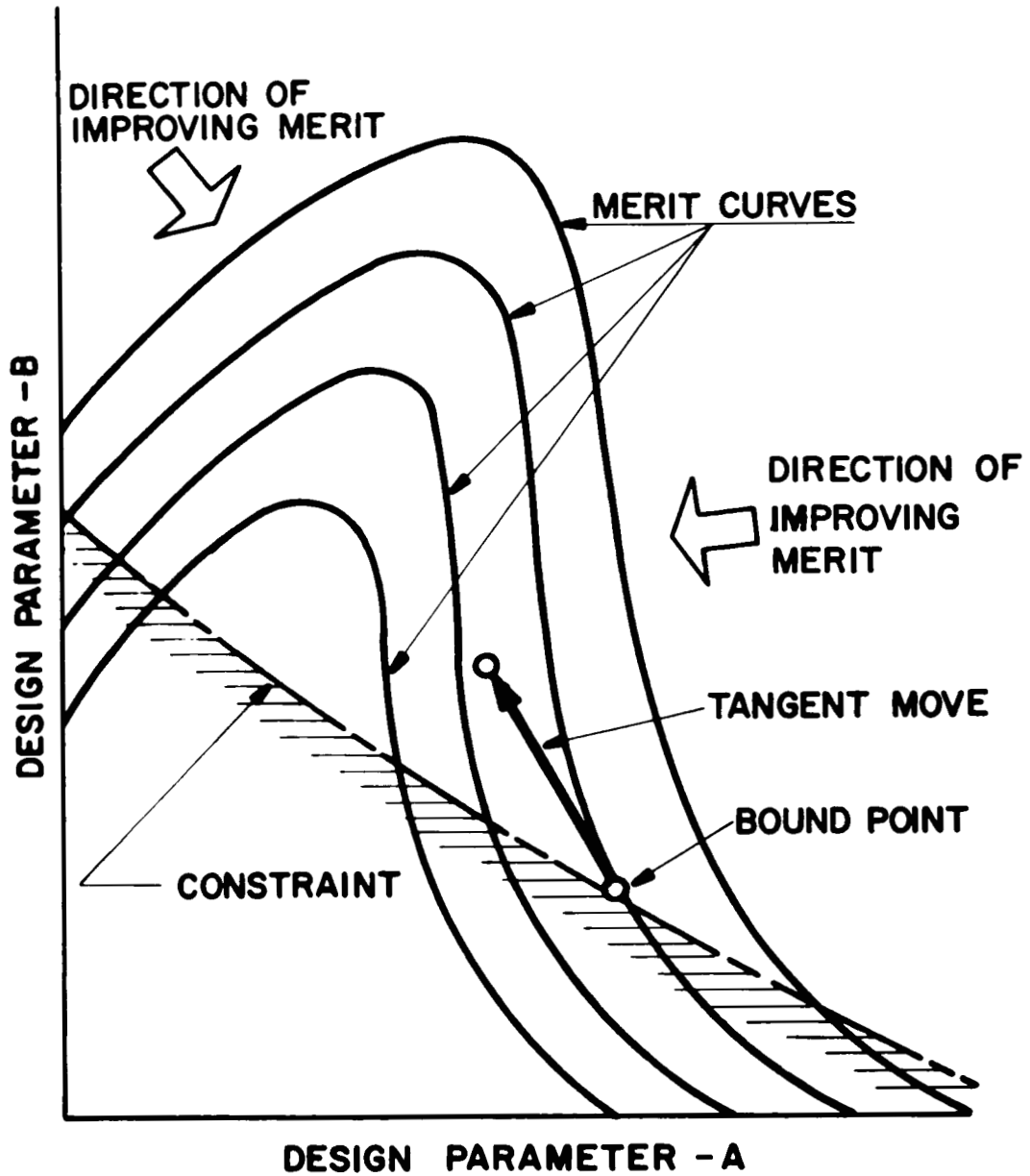
Figure 13. Concave Merit.
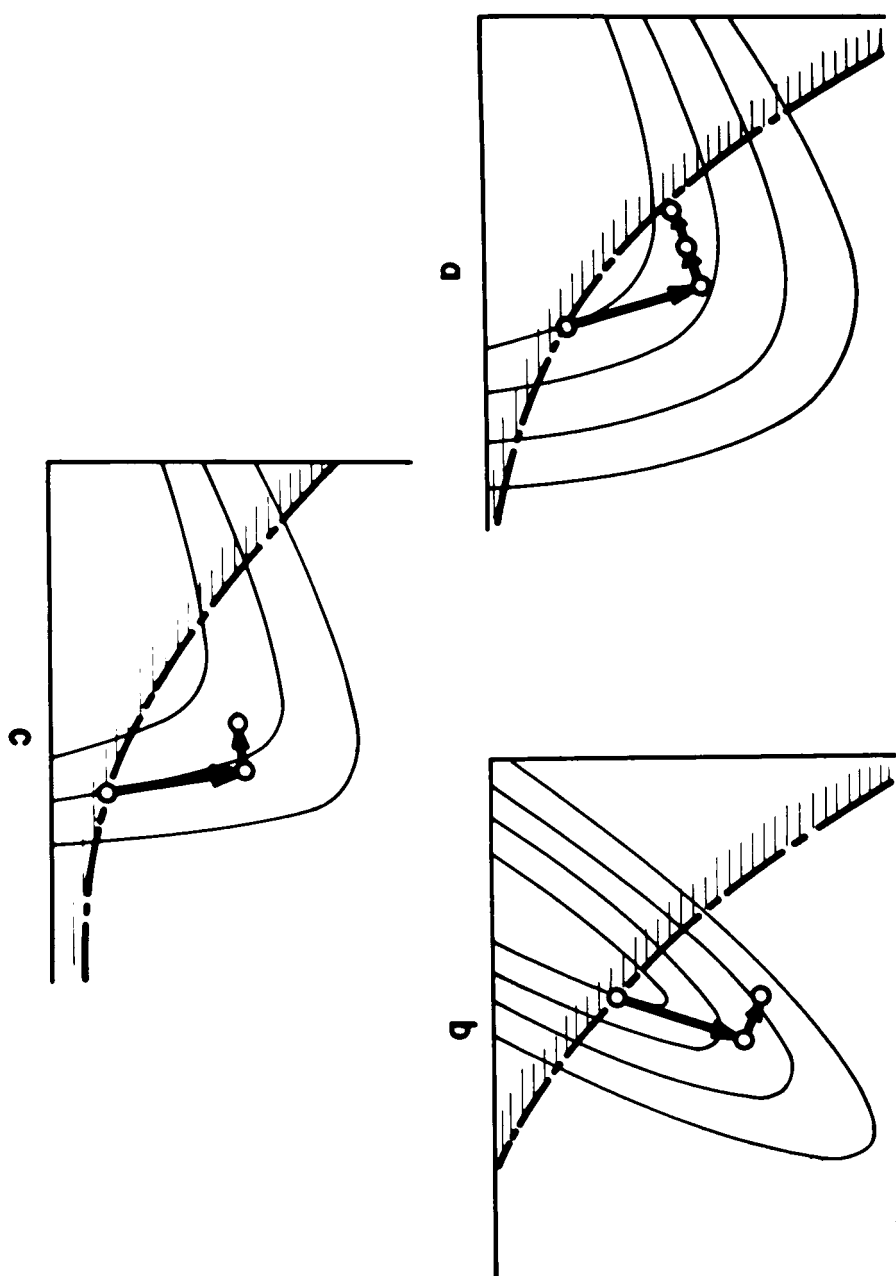
Figure 14. Tangent Possiblities.
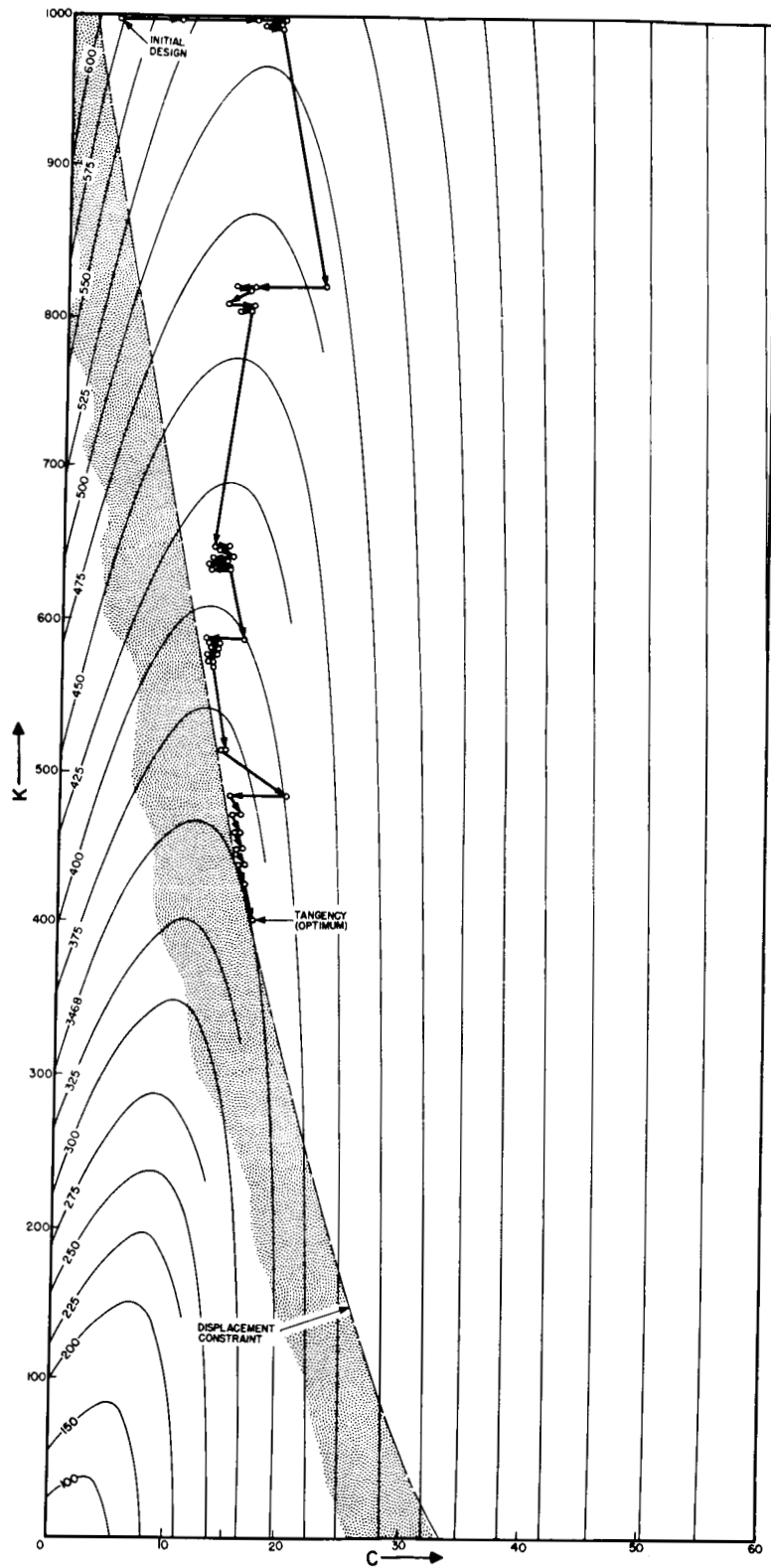
Figure 15. Design Path, Case A.
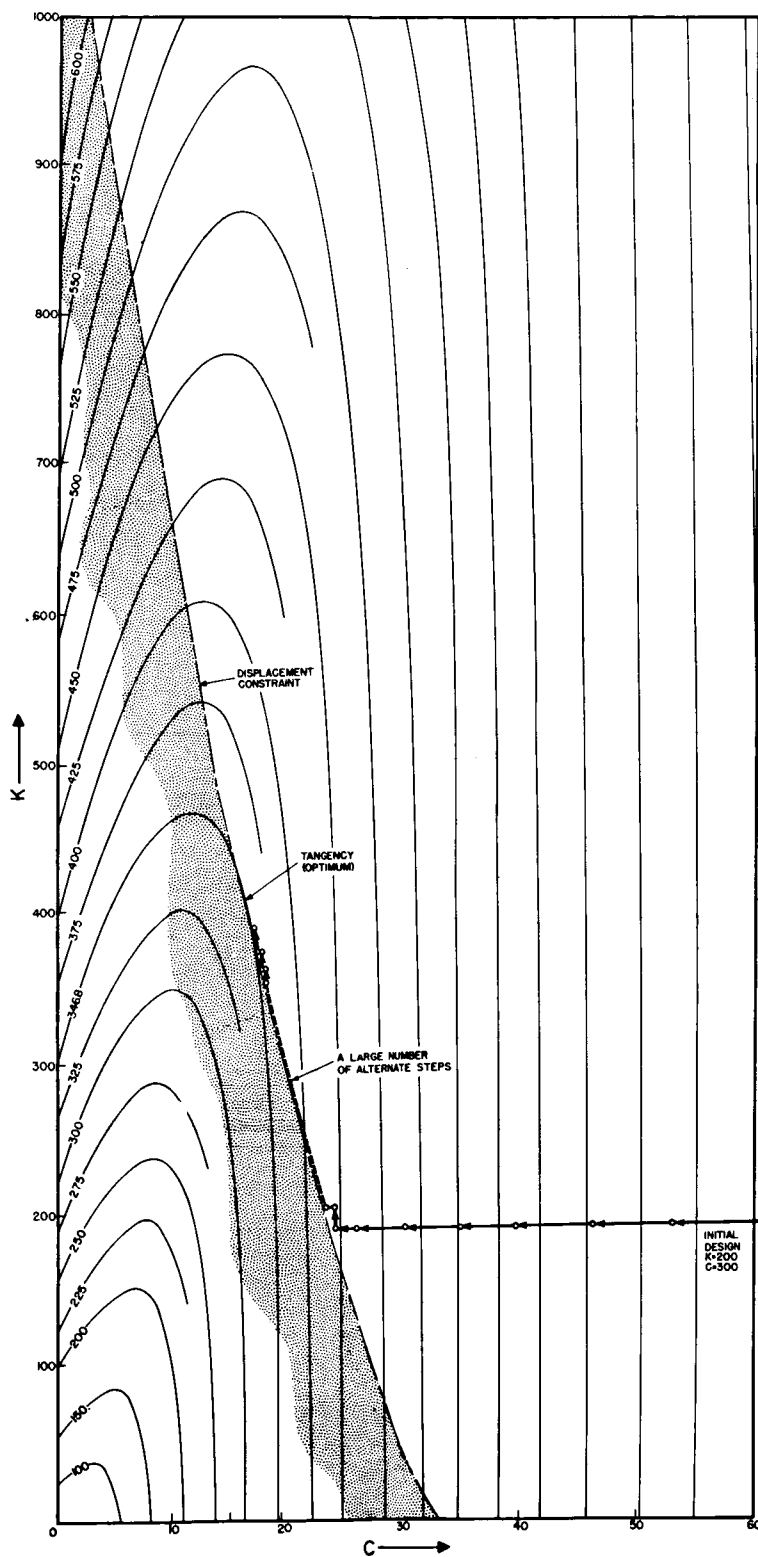
Figure 16. Design Path, Case B
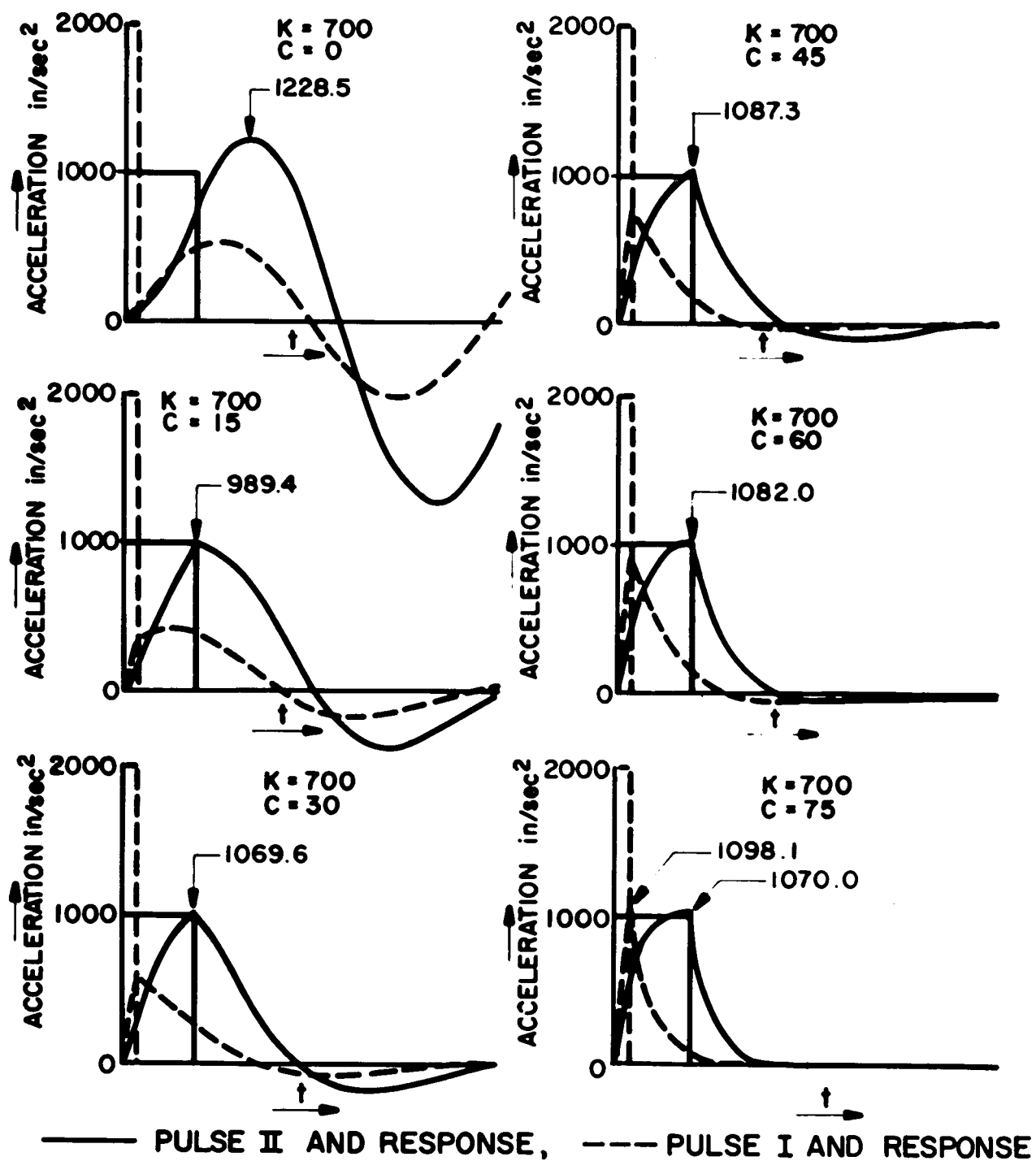
Figure 17. Acceleration Response.
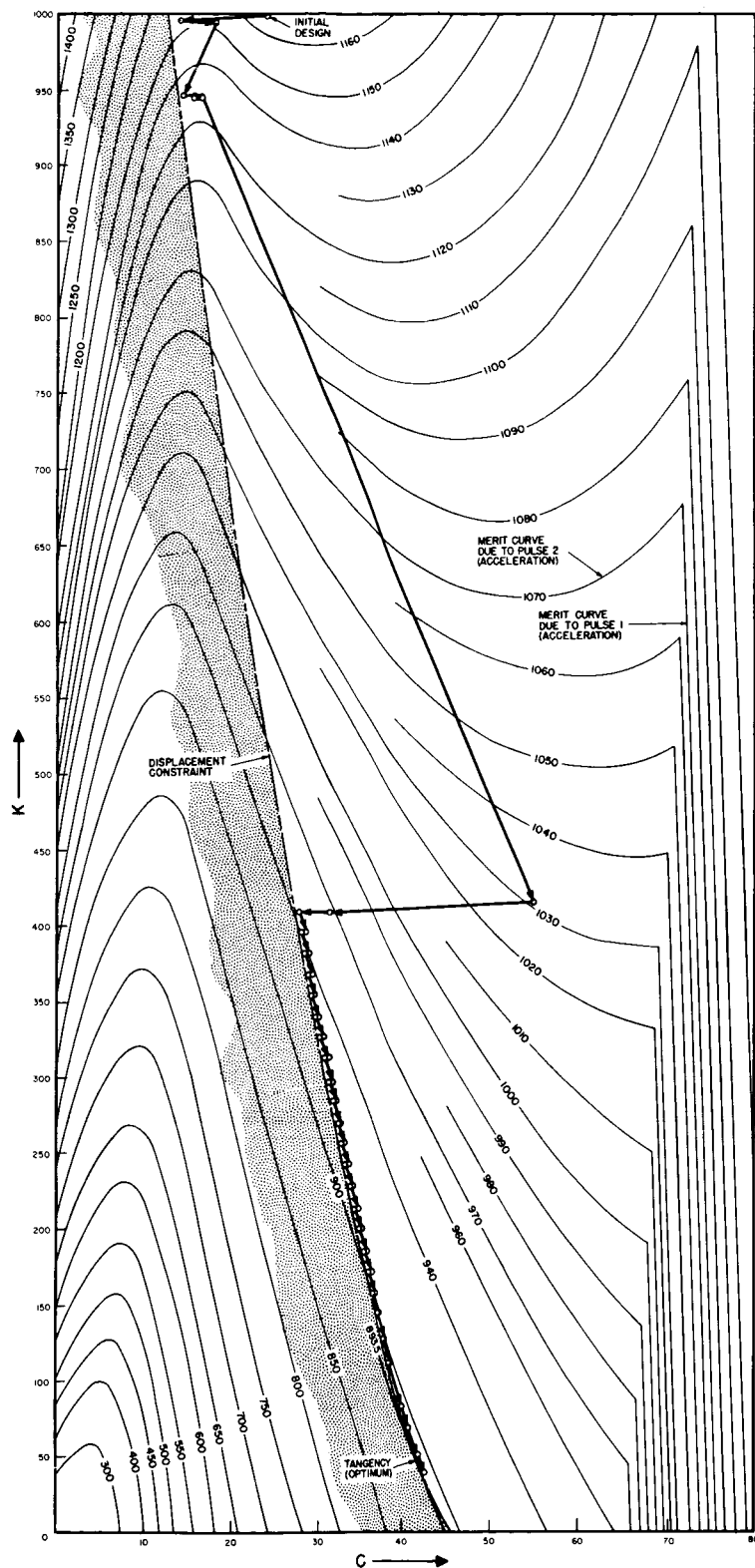
Figure 18. Design Path, Case C
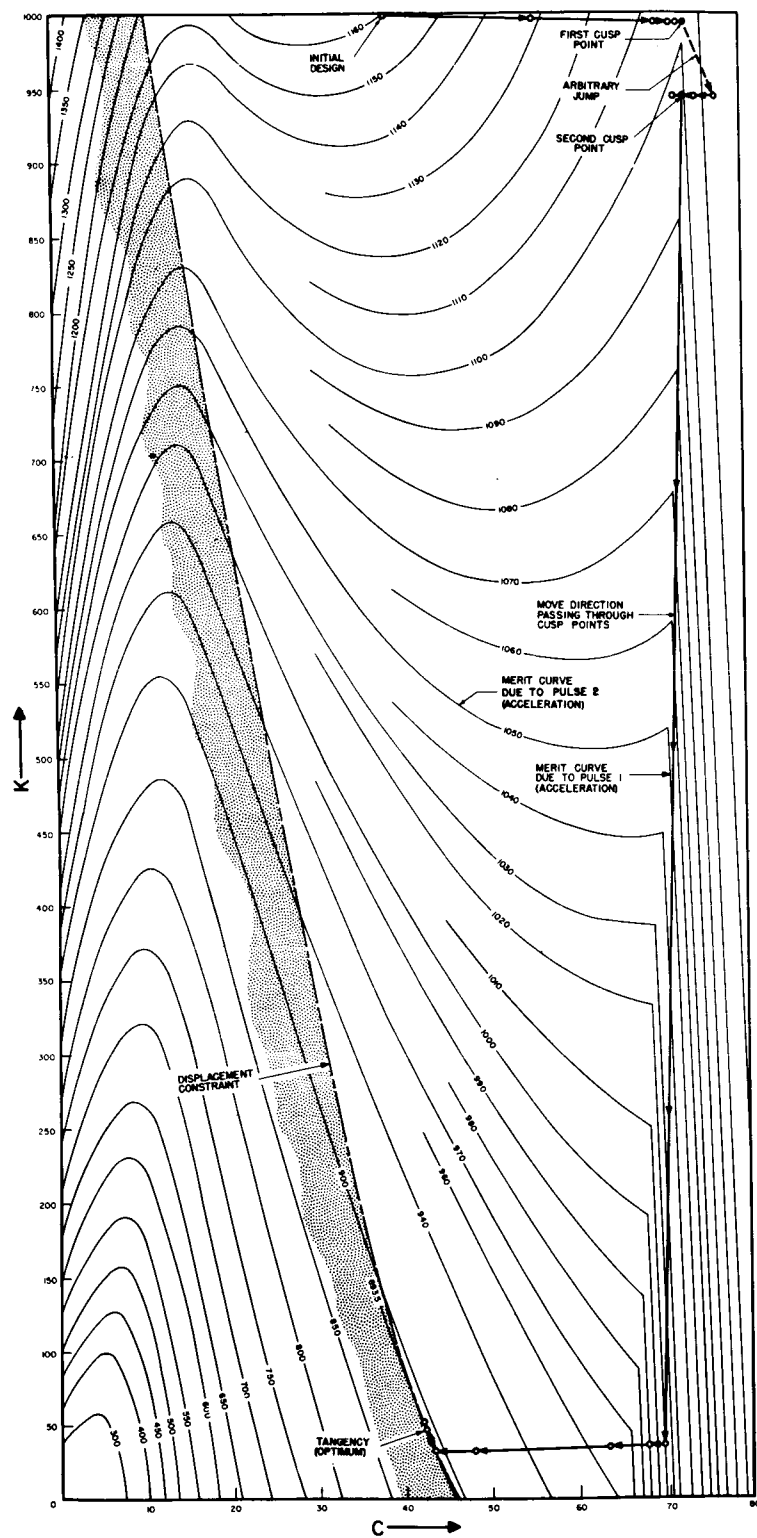
Figure 19. Design Path, Case D
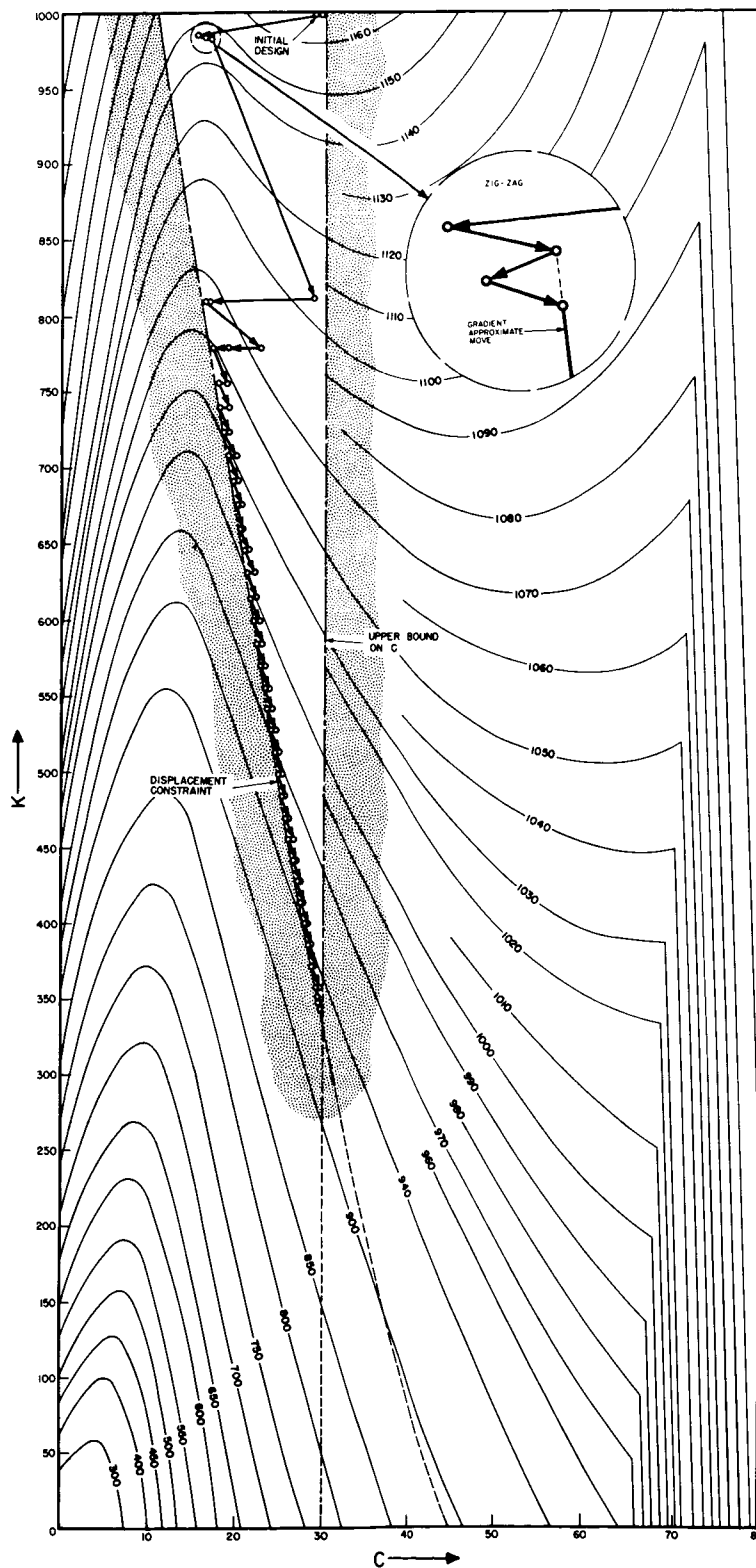
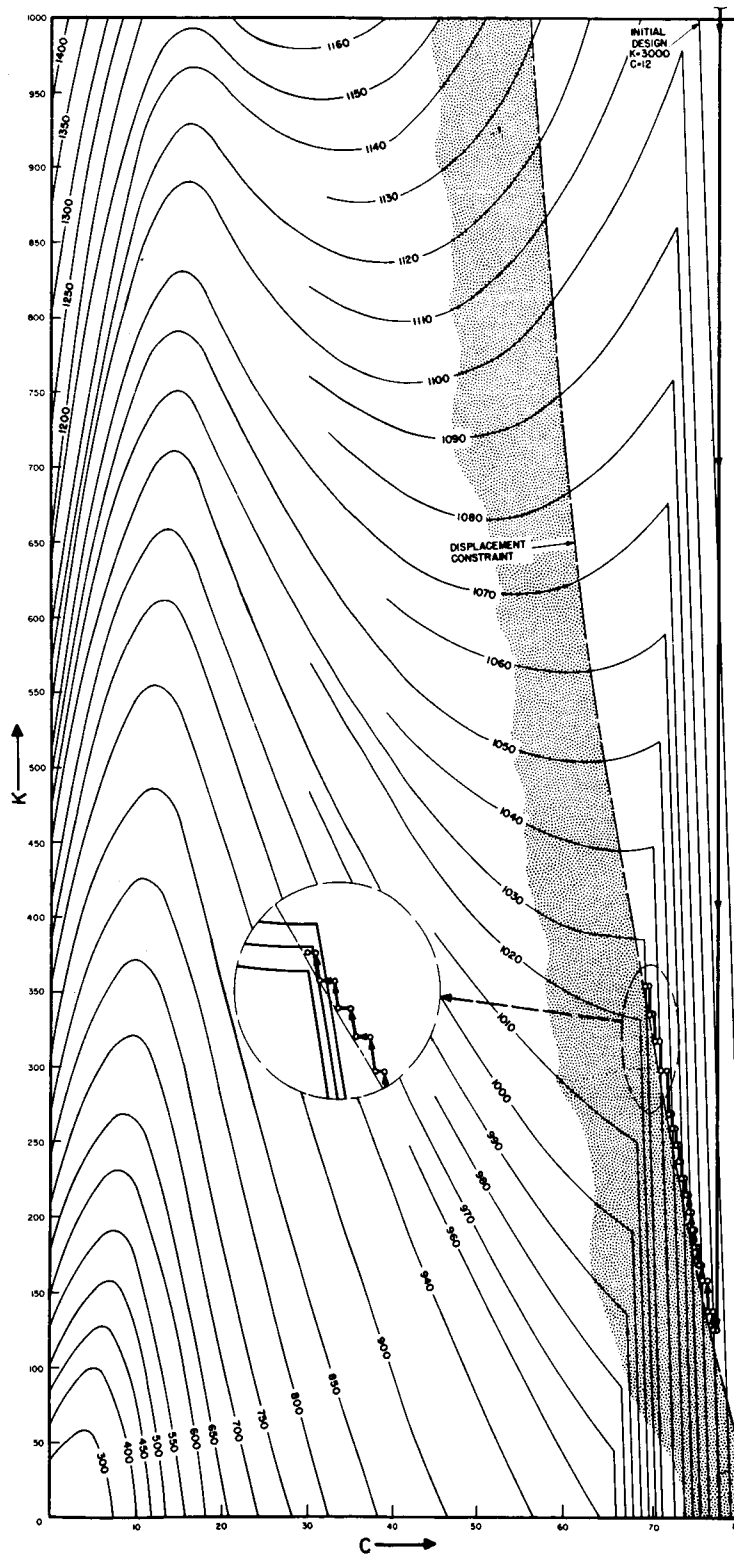Figure 20. Design Path, Case E.
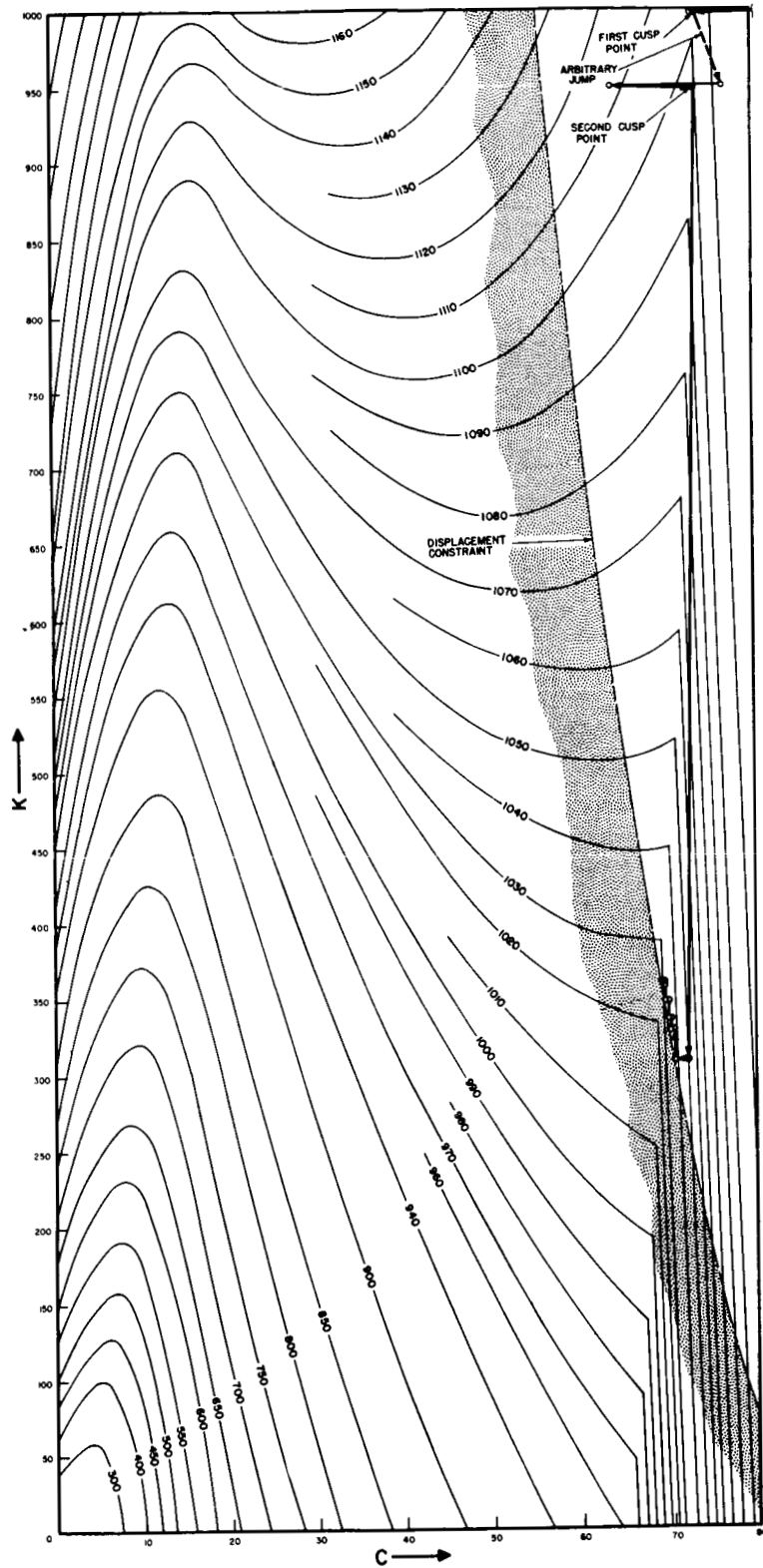
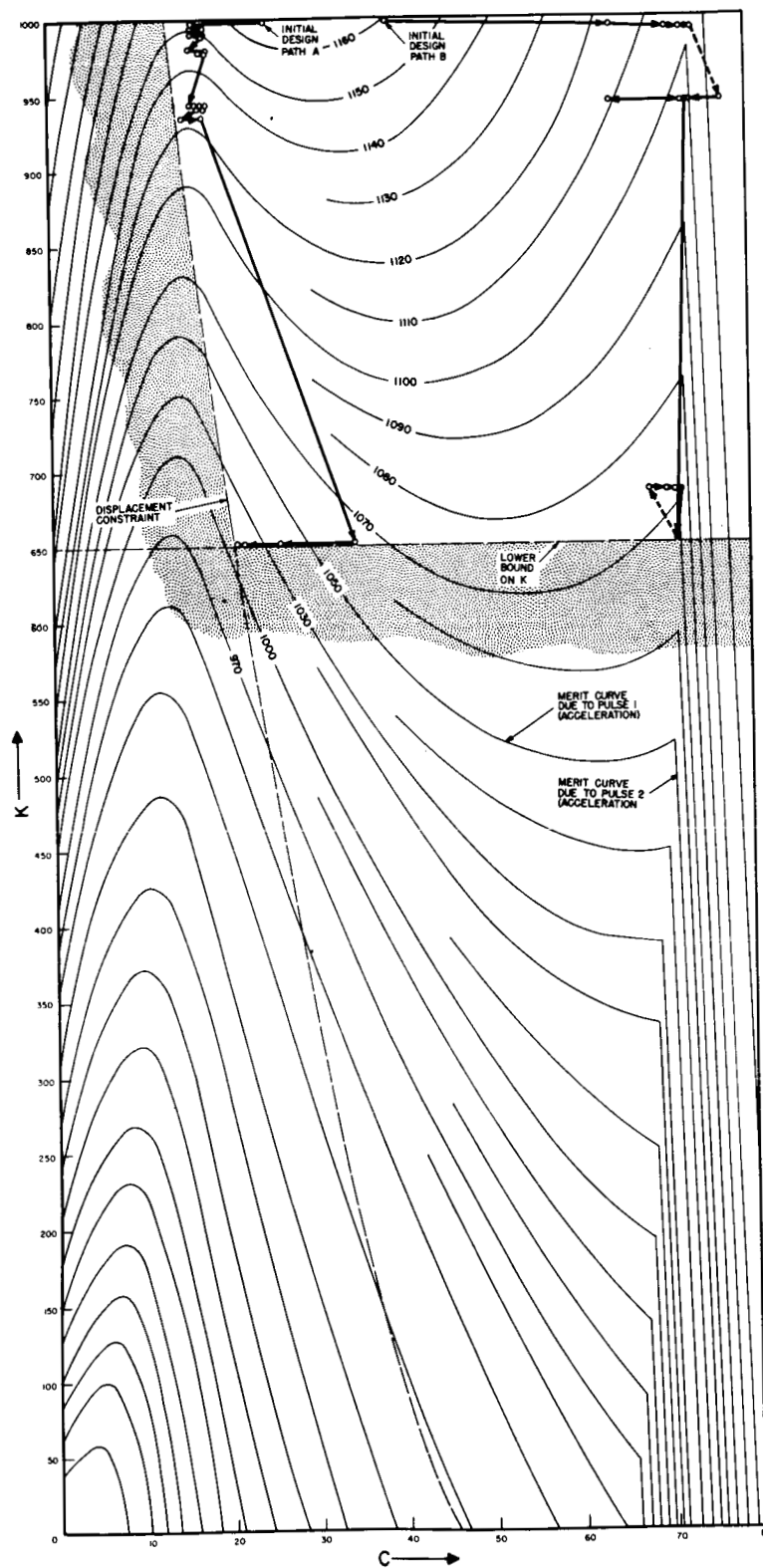Figure 21. Design Path, Case F.

Figure 22. Design Path, Case G.
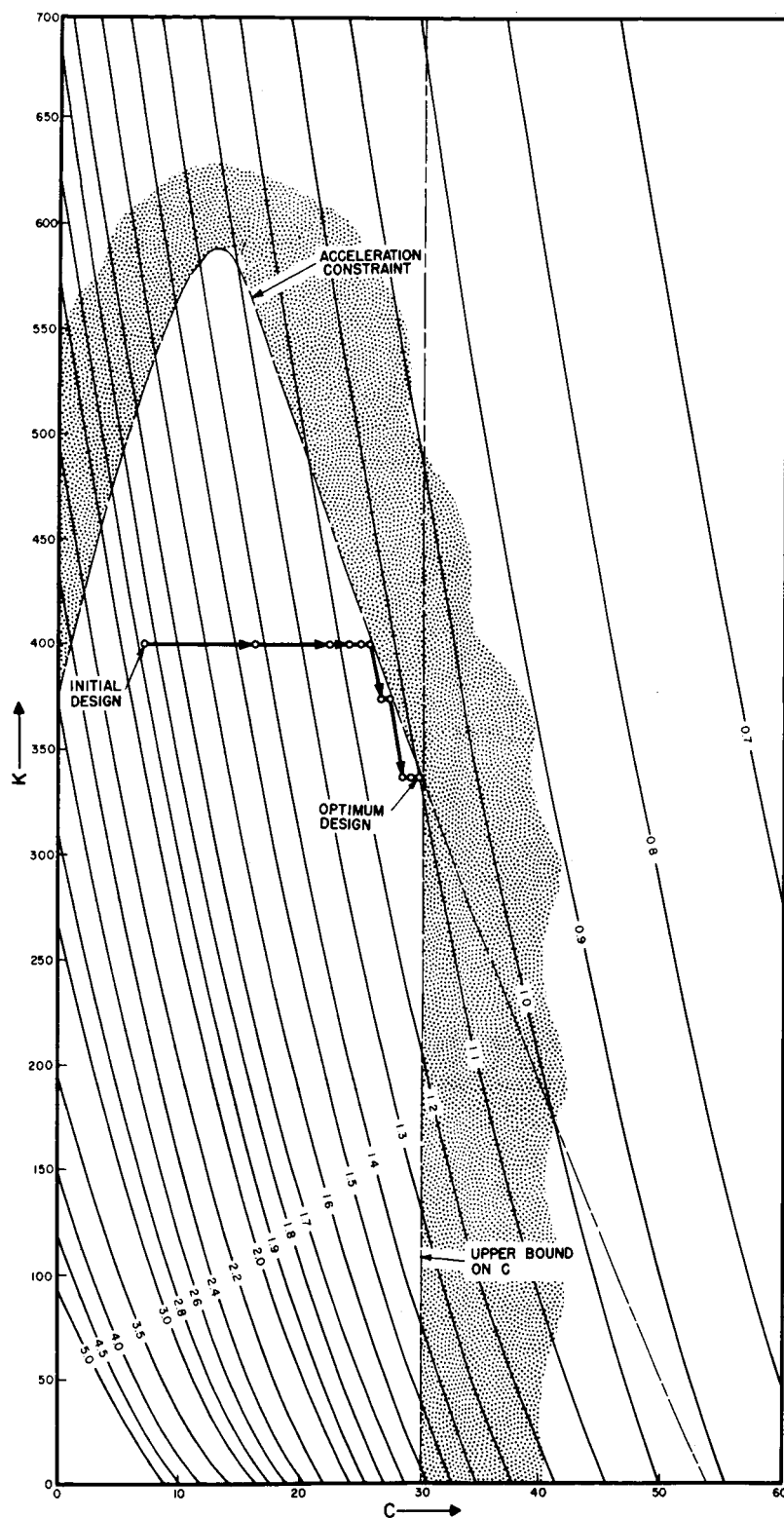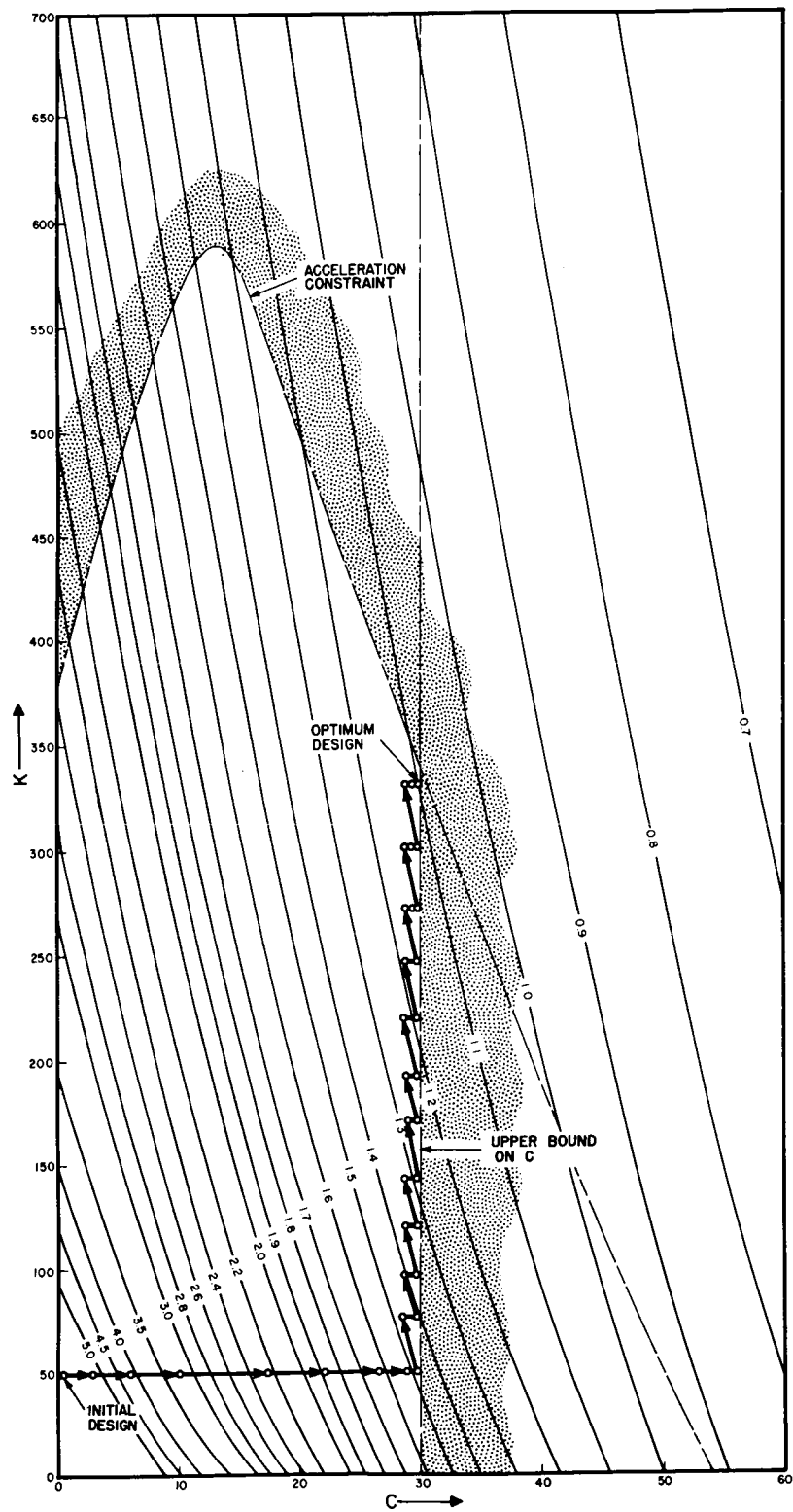
Figure 23.   Design Paths, Cases H and J.

Figure 24. Design Path, Case K.

Figure 25. Design Path, Case L.